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BIOMETRIKA

A JOURNAL FOR THE STATISTICAL STUDY OF BIOLOGICAL PROBLEMS

FOUNDED BY

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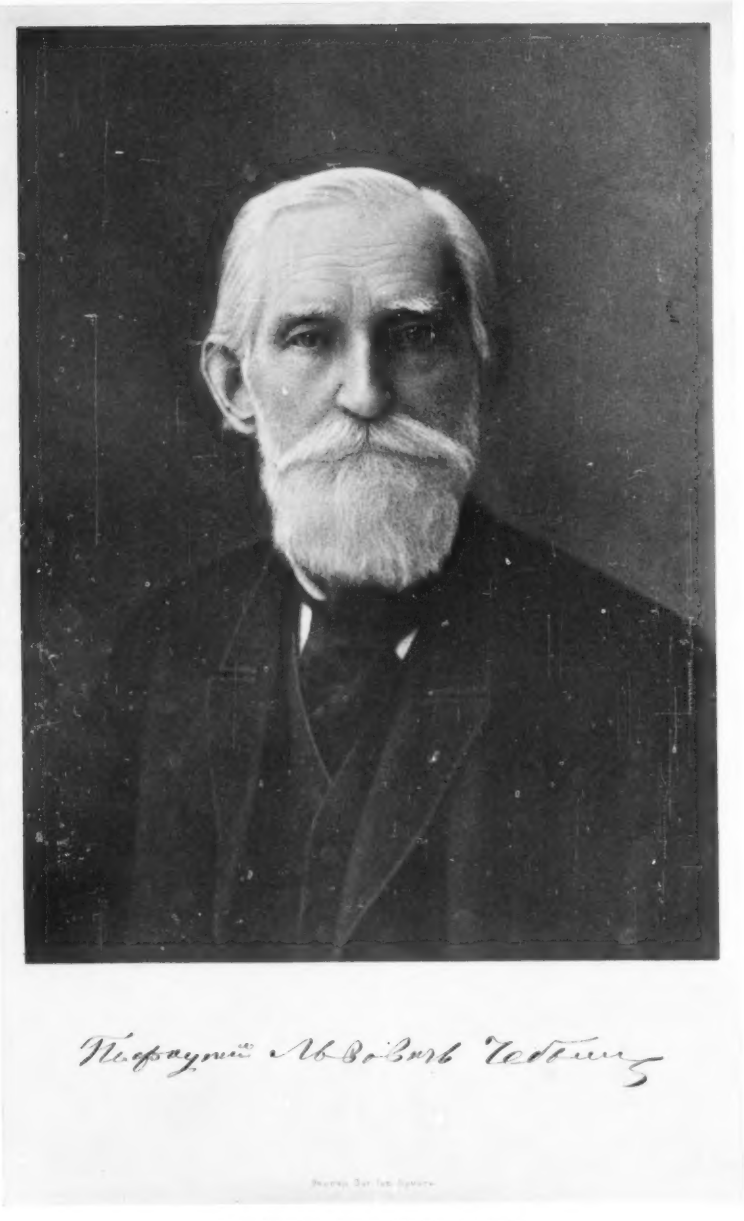

Tschebyschew, 1821—1894.

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VOLUME XXII JULY, 1930 PARTS I AND II

BIOMETRIKA

ON THE REMAINING TABLES FOR DETERMINING THE VOLUMES OF A BI-VARIATE NORMAL SURFACE.

EDITORIAL.

WE start with the fundamental tetrachoric table

a	b	$a+b$
c	d	$c+d$
$a+c$	$b+d$	N

and assume the frequency distribution to be normal; we suppose

$$\left.\begin{aligned}(b+d)/N &= \int_h^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = \tfrac{1}{2}(1-\alpha_h)\\ (c+d)/N &= \int_k^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = \tfrac{1}{2}(1-\alpha_k)\end{aligned}\right\} \quad \text{.....(i),}$$

and we take as our standard case h and k both positive. We can always arrange our table so that this shall be so. But having done this the correlation will sometimes be positive and sometimes negative.

The equation for r is known to be

$$\frac{d}{N} = \tau_0(h)\,\tau_0(k) + \tau_1(h)\,\tau_1(k)\,r + \tau_2(h)\,\tau_2(k)\,r^2 + \ldots + \tau_n(h)\,\tau_n(k)\,r^n + \ldots \quad \text{(ii)},$$

where τ_n is the tetrachoric function of the nth order, and $\tau_0(h) = \frac{1}{2}(1-\alpha_h)$, $\tau_0(k) = \frac{1}{2}(1-\alpha_k)$.

Tables of $\frac{d}{N}$ for the triple entry h, k, r, r being positive, have been published in *Biometrika**, and for $r = -\cdot80$ to $-1\cdot00$ in the same Journal†. Both these tables were computed by Dr Alice Lee. The present tables complete the whole series by providing the values of d/N from $r = \cdot00$ to $-\cdot75$. They have been computed by Margaret Moul, Ethel M. Elderton, E. C. Fieller, J. Pretorius and A. E. R. Church, all members of the Galton Laboratory. Up to $r = -\cdot60$ the values of d/N were obtained by aid of Dr Lee's table of the first twenty tetrachoric functions‡. After $r = -\cdot60$ it was found that twenty tetrachoric functions to only seven figures were not adequate and the integral value of d/N was obtained by quadrature, Weddle's formula being used, in the manner indicated in *Biometrika*, Vol. VIII.

* Vol. XIX. (1927), pp. 354—404.

† Vol. XI. pp. 284—291.

‡ *Biometrika*, Vol. XVII. pp. 343—354.

p. 386. The difference between the table there provided for $r = \cdot 80$ to $1 \cdot 00$ being that the present table is worked to more decimal places for the high values of h and k, those of the 1917 table having for certain cases been found inadequate.

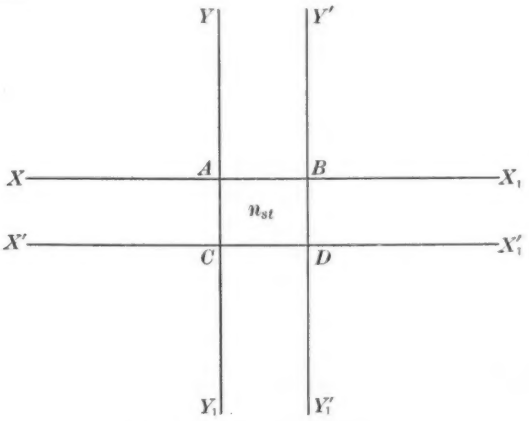
The complete tables thus furnished will serve three fundamental purposes: (i) to find r from any fourfold table, (ii) to find r from any cell of a table when the table is known or assumed to be normal in character, and (iii) to find when r has been ascertained for a table, for example by the product-moment method, what should be the theoretical contents of a given cell.

The general method of interpolating into tables of triple entry like the present has been discussed at adequate length in the paper of 1927 *. Examples of the use of these methods were provided, but it appeared that to work out effectively by these methods the contents of all the cells of a normal table we required the d/N tables for r negative. These are now supplied in association with the tables published in this Journal, Vol. XI. pp. 284—291.

It must be remembered that in our standard table we suppose d to be the contents of the quadrant for which the limits of integration are $x = h$ to ∞, $y = k$ to ∞, h and k being positive. It may be needful at times to find a, b or c from d, or on the contrary d from a, b or c. Since h and k are supposed known the connecting equations clearly are:

$$\left.\begin{aligned} \frac{a}{N} &= \tfrac{1}{2}(1+\alpha_h) - \tfrac{1}{2}(1-\alpha_k) + \frac{d}{N} \\ \frac{b}{N} &= \tfrac{1}{2}(1-\alpha_k) - \frac{d}{N} \\ \frac{c}{N} &= \tfrac{1}{2}(1-\alpha_k) - \frac{d}{N} \end{aligned}\right\} \quad \text{.....(iii).}$$

Now let n_{st} be the contents of the cell in the sth row and tth column of a correlation table.



* *Biometrika*, Vol. XIX. pp. 355—358.

Let n_u equal the total frequency or volume of the normal surface in the quadrant standing on YAX_1; n_u' that in the quadrant $Y'BX_1$; n_v that in the quadrant YCX_1'; and n_v' that in $Y'DX_1'$. Then $n_v - n_u = n_{st} + V$, where V is the volume standing on X_1BDX_1'. But $V = n_v' - n_u'$.

Accordingly $$n_{st} = n_v - n_u - n_v' + n_u' \quad \text{.....(iv).}$$

Now it is clear that the h_1, h_2 giving the lines YY_1 and $Y'Y_1'$, and the k_1, k_2 giving the lines XX_1 and $X'X_1'$, will be known; also r, the correlation coefficient, will be known. Thus either n_v, n_u, n_v', n_u' form the d's of four tetrachoric tables and are known, or, if they be the a, b or c's, the corresponding d can be obtained from the tables and their values found from Equations (iii) above.

Thus we deduce the "normal value" of n_{st}. We propose first to illustrate this process.

Illustration I. In a table for the correlation of Father and Son for stature we find, for the heights of Fathers 68″·875—69″·875, twelve Sons of the heights 66″·875—67″·875. This is a perfectly arbitrary cell taken out of a table of 20×17 cells*. The correlation coefficient of this table worked by the product-moment method is ·5189. The problem we put before ourselves is this: Supposing the table corresponds to a normal surface, are twelve individuals a reasonable frequency for this cell? As much of the table as concerns our present purpose can be written as follows:

Sons' Stature	Fathers' Stature			Totals
	Below 68″·875	68″·875—69″·875	Above 69″·875	
Below 66″·875	206	9	10	225
66″·875—67″·875	105	[12]	12	129
Above 67″·875	326	104	216	646
Totals	637	125	238	1000

Clearly $$n_u = 19, \quad n_u' = 10, \quad n_v = 43, \quad n_v' = 22,$$

and $$n_{st} = 12 = n_v - n_u - n_v' + n_u' = 43 - 19 - 22 + 10.$$

We can now examine the requisite four tables which have to be solved to obtain n_{st} for the normal surface. They are:

(i)			(ii)			(iii)			(iv)		
	(n_u)			(n_u')			(n_v)			(n_v')	
206	19	225	215	10	225	311	43	354	332	22	354
431	344	775	547	228	775	326	320	646	430	216	646
637	363	1000	762	238	1000	637	363	1000	762	238	1000

* See *Biometrika*, Vol. XIV. p. 151, Table XV.

If we re-arrange these tables in standard form, we have:

(i)			(ii)			(iii)			(iv)		
(a_1)	(b_1)		(a_2)	(b_2)		(a_3)	(b_3)		(a_4)	(b_4)	
431	344	775	547	228	775	326	320	646	430	216	646
(c_1)	$(d_1=n_u)$		(c_2)	$(d_2=n_u')$		(c_3)	$(d_3=n_v)$		(c_4)	$(d_4=n_v')$	
206	19	225	215	10	225	311	43	354	332	22	354
637	363	1000	762	238	1000	637	363	1000	762	238	1000

and we see at once that $ad-bc$ is *negative* for all of them, or the d/N is to be found from the present issue of tables; i.e. $r=-\cdot5189$. In the next place in *every* case the n_u, n_u', n_v, n_v' of the quadrant to be found is the d of the standard form. Hence we have, by Equation (iv),

$$n_{st}=N\left(\frac{d_3}{N}-\frac{d_1}{N}-\frac{d_4}{N}+\frac{d_2}{N}\right),$$

where the d/N's are to be found from our present table. To use these, however, we require to ascertain the h and k corresponding to the above four tables. This is most easily done by the use of the first and last columns in Table XXIX of the *Tables for Statisticians*, Part I, which give h (or k) for $\frac{1}{2}(1-\alpha)$. In the present case we have:

$\frac{1}{2}(1-\alpha_{h_1})=\cdot363$, $\frac{1}{2}(1-\alpha_{k_1})=\cdot225$, or: $h_1=\cdot35045$, $k_1=\cdot75541$,

$\frac{1}{2}(1-\alpha_{h_2})=\cdot238$, $\frac{1}{2}(1-\alpha_{k_2})=\cdot225$, or: $h_2=\cdot71275$, $k_2=\cdot75541$,

$\frac{1}{2}(1-\alpha_{h_3})=\cdot363$, $\frac{1}{2}(1-\alpha_{k_3})=\cdot354$, or: $h_3=\cdot35045$, $k_3=\cdot37454$,

$\frac{1}{2}(1-\alpha_{h_4})=\cdot238$, $\frac{1}{2}(1-\alpha_{k_4})=\cdot354$, or: $h_4=\cdot71275$, $k_4=\cdot37454$.

For most cases h and k to five decimal figures are fully adequate*.

If the four tables be now worked out by the interpolation formula for (i) use of four entries, and (ii) for twelve entries (i.e. formulae (α) and (β) of *Biometrika*, Vol. XIX. p. 356), we find:

	d_1	d_2	d_3	d_4
(β)	27·113	12·752	56·437	28·348
(α)	27·313	12·847	56·674	28·467
Observed values:	19	10	43	22

In both cases linear interpolation alone has been used to deduce $r=-\cdot5189$ from the tables for $r=-\cdot50$ and $r=-\cdot55$, after the readings have been obtained for these from the corresponding h's and k's either by (α) or (β). It will be seen at once that (α) and (β) are in very close agreement, and that, at any rate in this portion of the tables, the hyperbolic formula (α) is fully adequate for most practical statistical purposes.

* A table of h to $\frac{1}{2}(1-\alpha_h)$ with far more figures will shortly be published.

But the deviations from the observed values of d in the four cases are very considerable. Notwithstanding, if we proceed to determine n_{st} we have:

$$\text{from } (\beta): \quad n_{st} = d_3 - d_1 - d_4 + d_2$$
$$= 56{\cdot}437 - 27{\cdot}113 - 28{\cdot}348 + 12{\cdot}752 = 13{\cdot}728,$$
$$\text{from } (\alpha): \quad n_{st} = 56{\cdot}674 - 27{\cdot}313 - 28{\cdot}467 + 12{\cdot}847 = 13{\cdot}741,$$

or (β) only improves on (α) by ·013, a quantity of no practical importance.

Now the standard error of 13·74 in 1000 $= \sqrt{\dfrac{13{\cdot}74 \times 986{\cdot}27}{1000}} = 3{\cdot}68$ nearly, corresponding to a probable error of 2·48.

Clearly $13{\cdot}74 \pm 2{\cdot}48$ easily covers the probability of 12 arising in a random sample. Or, the observed cell content of 12 is quite consistent with the table for the correlation of father's and son's statures being of a normal type.

Illustration II. We will take another example from the same correlation table, which indicates a greater variety in the methods of treatment; namely, the cell for fathers of stature 67″·875—68″·875 and for sons 67″·875—68″·875. It contains 27 cases, and the full table condensed for our purposes is as follows:

Sons' Stature	Fathers' Stature			Totals
	Below 67″·875	67″·875—68″·875	Above 68″·875	
Below 67″·875	277	34	43	354
67″·875—68″·875	89	27	65	181
Above 68″·875	132	78	255	465
Totals	498	139	363	1000

We have at once:

$$n_u = 77, \quad n_u' = 43, \quad n_v = 169, \quad n_v' = 108.$$

Thus our four tables take the forms:

(i)			(ii)			(iii)			(iv)		
277	(n_u) 77	354	311	(n_u') 43	354	366	(n_v) 169	535	427	(n_v') 108	535
221	425	646	326	320	646	132	333	465	210	255	465
498	502	1000	637	363	1000	498	502	1000	637	363	1000

Or, arranged in standard form :

(i)			(ii)		
$a_1=425$	$b_1=221$	646	$a_2=326$	$b_2=320$	646
(n_u) $c_1=77$	$d_1=277$	354	$c_2=311$	(n_u') $d_2=43$	354
502	498	1000	637	363	1000

(iii)			(iv)		
(n_v) $a_3=169$	$b_3=366$	535	$a_4=427$	(n_v') $b_4=108$	535
$c_3=333$	$d_3=132$	465	$c_4=210$	$d_4=255$	465
502	498	1000	637	363	1000

We see at once that :

$n_u = c_1$, and r is positive in Table (i) $= +\cdot5189$,

$n_u' = d_2$, and r is negative in Table (ii) $= -\cdot5189$,

$n_v = a_3$, and r is negative in Table (iii) $= -\cdot5189$,

$n_v' = b_4$, and r is positive in Table (iv) $= +\cdot5189$.

We have thus : $$n_{st} = N\left(\frac{a_3}{N} - \frac{c_1}{N} - \frac{b_4}{N} + \frac{d_2}{N}\right).$$

There are thus two tables to be worked from the tables in *Biometrika*, Vol. XIX. pp. 373—404, i.e. (i) and (iv), and two tables from the present tables, i.e. (ii) and (iii).

Accordingly, in only one case is the n_u or n_v equal to d, namely $n_u' = d_2$. For the other cases we require to use the formulae given in Equations (iii).

Further, we shall need to use special interpolation formulae for three of the cases, as we are at the edges of our tables for d/N. We may arrange our work as shown on the following page, where α, β, γ, $\gamma\,bis$, and δ refer to the formulae in *Biometrika*, Vol. XIX. pp. 356—358.

It is, we think, clear that a difference of the order 0·223 is not of much statistical importance in a cell containing 28, and thus the formula α might have been used throughout. We give the work up to third differences, which much increases the labour, in order to show the reasonable effectiveness of the shorter hyperbolic formula.

(i)

$\frac{1}{2}(1-\alpha_h) = \cdot 498$, $\frac{1}{2}(1-\alpha_k) = \cdot 354$

$h = \cdot 00501$, $k = \cdot 37454$

Finial panel in h

Use Formula γ *bis*.

$\theta = \cdot 0501$,	$\chi = \cdot 7454$
$\phi = \cdot 9499$,	$\psi = \cdot 2546$
$\phi\psi = \cdot 24184$,	$\phi\chi = \cdot 70806$
$\theta\psi = \cdot 01276$,	$\theta\chi = \cdot 03734$
$\frac{1}{6}\theta\phi = \cdot 00793$,	$\frac{1}{6}\chi\psi = \cdot 03163$

$r = \cdot 50$

$z_{00} = \cdot 270,344$,	$z_{01} = \cdot 248,589$
$z_{10} = \cdot 255,392$,	$z_{11} = \cdot 235,345$
$\delta^2 z_{10} = -722$,	$\delta'^2 z_{00} = -213$
$\delta^2 z_{11} = -704$,	$\delta'^2 z_{10} = -278$
$\delta^2 z_{20} = -556$,	$\delta'^2 z_{01} = +44$
$\delta^2 z_{21} = -561$,	$\delta'^2 z_{11} = -38$

$\phi\psi z_{00} + \phi\chi z_{01} + \theta\psi z_{10} + \theta\chi z_{11} = \cdot 2534,4250$

$-\frac{1}{6}\theta\phi\ (4+\phi)\ (\psi\delta^2 z_{10} + \chi\delta^2 z_{11}) + 2781$

$+\frac{1}{6}\theta\phi\ (1+\phi)\ (\psi\delta^2 z_{20} + \chi\delta^2 z_{21}) - 865$

$-\frac{1}{6}\psi\chi\ (1+\psi)\ (\phi\delta'^2 z_{00} + \theta\delta'^2 z_{10}) + 858$

$-\frac{1}{6}\chi\psi\ (1+\chi)\ (\phi\delta'^2 z_{01} + \theta\delta'^2 z_{11}) - 220$

$d_1/N = \cdot 2534,6804$

$r = \cdot 55$

$z_{00} = \cdot 279,133$,	$z_{01} = \cdot 256,963$
$z_{10} = \cdot 264,313$,	$z_{11} = \cdot 243,907$
$\delta^2 z_{10} = -845$,	$\delta'^2 z_{00} = -319$
$\delta^2 z_{11} = -821$,	$\delta'^2 z_{10} = -391$
$\delta^2 z_{20} = -678$,	$\delta'^2 z_{01} = -46$
$\delta^2 z_{21} = -679$,	$\delta'^2 z_{11} = -137$

The five several contributions to $z_{\theta,\chi}$ are:

$\cdot 2619,3087$
$+ \quad 3247$
$- \quad 1050$
$+ \quad 1280$
$+ \quad 279$
$\cdot 2619,6843$

$$\frac{d_1}{N} = \cdot 2534,6804 + \frac{189}{500}(850039)$$

$$= \cdot 2566,8118$$

$$d_1 = 256 \cdot 681$$

If we used only the hyperbolic terms:

$d_1 = 256 \cdot 651$

But $n_u = c_1$

$= 354 - 256 \cdot 681$

$= 97 \cdot 319$ from γ *bis*

$= 354 - 256 \cdot 651$

$= 97 \cdot 349$ from α

(ii)

$\frac{1}{2}(1-\alpha_h) = \cdot 363$, $\frac{1}{2}(1-\alpha_k) = \cdot 354$

$h = \cdot 35045$, $k = \cdot 37454$

Neither h nor k finial

Use Formula β.

$\theta = \cdot 5045$,	$\chi = \cdot 7454$
$\phi = \cdot 4955$,	$\psi = \cdot 2546$
$\phi\psi = \cdot 12615$,	$\phi\chi = \cdot 36935$
$\theta\psi = \cdot 12845$,	$\theta\chi = \cdot 37605$
$\frac{1}{6}\theta\phi = \cdot 04166$,	$\frac{1}{6}\chi\psi = \cdot 03163$

$r = -\cdot 50$

$z_{00} = \cdot 072,4876$,	$z_{01} = \cdot 061,5434$
$z_{10} = \cdot 061,5434$,	$z_{11} = \cdot 052,0367$
$\delta^2 z_{00} = 11101$,	$\delta'^2 z_{00} = 11101$
$\delta^2 z_{01} = 10164$,	$\delta'^2 z_{10} = 10164$
$\delta^2 z_{10} = 11309$,	$\delta'^2 z_{01} = 11309$
$\delta^2 z_{11} = 10248$,	$\delta'^2 z_{11} = 10248$

$\phi\psi z_{00} + \phi\chi z_{01} + \theta\psi z_{10} + \theta\chi z_{11} = \cdot 0593,4901$

$-\frac{1}{6}\theta\phi\ (1+\phi)\ (\psi\delta^2 z_{00} + \chi\delta^2 z_{01}) - 6481$

$-\frac{1}{6}\theta\phi\ (1+\theta)\ (\psi\delta^2 z_{10} + \chi\delta^2 z_{11}) - 6592$

$-\frac{1}{6}\chi\psi\ (1+\psi)\ (\phi\delta'^2 z_{00} + \theta\delta'^2 z_{10}) - 4218$

$-\frac{1}{6}\chi\psi\ (1+\chi)\ (\phi\delta'^2 z_{01} + \theta\delta'^2 z_{11}) - 5948$

$d_2/N = \cdot 0591,1662$

$r = -\cdot 55$

$z_{00} = \cdot 064,7508$,	$z_{01} = \cdot 054,3306$
$z_{10} = \cdot 054,3306$,	$z_{11} = \cdot 064,3610$
$\delta^2 z_{00} = 11859$,	$\delta'^2 z_{00} = 11859$
$\delta^2 z_{01} = 10801$,	$\delta'^2 z_{10} = 10801$
$\delta^2 z_{10} = 11876$,	$\delta'^2 z_{01} = 11876$
$\delta^2 z_{11} = 10696$,	$\delta'^2 z_{11} = 10696$

The five several contributions to $z_{\theta,\chi}$ are:

$\cdot 0522,7209$
$- \quad 6897$
$- \quad 6892$
$- \quad 4494$
$- \quad 6228$
$\cdot 0520,2698$

$$\frac{d_2}{N} = \cdot 0591,1662 - \frac{189}{500}(708967)$$

$$= \cdot 0564,3672$$

$$d_2 = 56 \cdot 437$$

If we used only the hyperbolic terms:

$d_2 = 56 \cdot 674$

But $n_u' = d_2$

$= 56 \cdot 437$

from β

$= 56 \cdot 674$

from α

(iii)

$\frac{1}{2}(1-\alpha_h) = \cdot 498$, $\frac{1}{2}(1-\alpha_k) = \cdot 465$

$h = \cdot 00501$, $k = \cdot 08784$

Doubly finial panel

Use Formula δ.

$\theta = \cdot 0501$,	$\chi = \cdot 8784$
$\phi = \cdot 9499$,	$\psi = \cdot 1216$
$\phi\psi = \cdot 11551$,	$\phi\chi = \cdot 83439$
$\theta\psi = \cdot 00609$,	$\theta\chi = \cdot 04401$
$\frac{1}{6}\theta\phi = \cdot 00793$,	$\frac{1}{6}\chi\psi = \cdot 01780$

$r = -\cdot 50$

$z_{00} = \cdot 166,6667$,	$z_{01} = \cdot 147,2109$
$z_{10} = \cdot 147,2109$,	$z_{11} = \cdot 129,5818$
$\delta^2 z_{10} = 10992 = \delta'^2 z_{01}$	
$\delta^2 z_{11} = 10689 = \delta'^2 z_{11}$	
$\delta^2 z_{20} = 12466 = \delta'^2 z_{02}$	
$\delta^2 z_{21} = 11936 = \delta'^2 z_{12}$	

$\phi\psi z_{00} + \phi\chi z_{01} + \theta\psi z_{10} + \theta\chi z_{11} = \cdot 1486,8238$

$-\frac{1}{6}\theta\phi\ (4+\phi)\ (\psi\delta^2 z_{10} + \chi\delta^2 z_{11}) - 4210$

$+\frac{1}{6}\theta\phi\ (1+\phi)\ (\psi\delta^2 z_{20} + \chi\delta^2 z_{21}) + 1856$

$-\frac{1}{6}\psi\chi\ (4+\psi)\ (\phi\delta'^2 z_{01} + \theta\delta'^2 z_{11}) - 8053$

$+\frac{1}{6}\psi\chi\ (1+\psi)\ (\phi\delta'^2 z_{02} + \theta\delta'^2 z_{12}) + 2484$

$d_3/N = \cdot 1486,0315$

$r = -\cdot 55$

$z_{00} = \cdot 157,3139$,	$z_{01} = \cdot 137,9225$
$z_{10} = \cdot 137,9225$,	$z_{11} = \cdot 120,4240$
$\delta^2 z_{10} = 12254 = \delta'^2 z_{01}$	
$\delta^2 z_{11} = 11911 = \delta'^2 z_{11}$	
$\delta^2 z_{20} = 13648 = \delta'^2 z_{02}$	
$\delta^2 z_{21} = 13054 = \delta'^2 z_{12}$	

The five several contributions to $z_{\theta,\chi}$ are:

$\cdot 1393,9229$
$- \quad 4692$
$+ \quad 2030$
$- \quad 8977$
$+ \quad 2719$
$\cdot 1393,0309$

$$\frac{d_3}{N} = \cdot 1486,0315 - \frac{189}{500}(930006)$$

$$= \cdot 1450,8773$$

$$d_3 = 145 \cdot 088$$

If we used only the hyperbolic terms:

$d_3 = 145 \cdot 171$

But $n_v = a_3$

$= 502 - 465 + 145 \cdot 088$

$= 182 \cdot 088$ from δ

$= 182 \cdot 171$

from α

(iv)

$\frac{1}{2}(1-\alpha_h) = \cdot 363$, $\frac{1}{2}(1-\alpha_k) = \cdot 465$

$h = \cdot 35045$, $k = \cdot 08784$

Finial panel in k

Use Formula γ.

$\theta = \cdot 5045$,	$\chi = \cdot 8784$
$\phi = \cdot 4955$,	$\psi = \cdot 1216$
$\phi\psi = \cdot 06025$,	$\phi\chi = \cdot 43525$
$\theta\psi = \cdot 06135$,	$\theta\chi = \cdot 44315$
$\frac{1}{6}\theta\phi = \cdot 04166$,	$\frac{1}{6}\chi\psi = \cdot 01780$

$r = \cdot 50$

$z_{00} = \cdot 270,344$,	$z_{01} = \cdot 255,392$
$z_{10} = \cdot 248,589$,	$z_{11} = \cdot 235,345$
$\delta^2 z_{00} = -213$,	$\delta'^2 z_{01} = -722$
$\delta^2 z_{01} = -278$,	$\delta'^2 z_{11} = -704$
$\delta^2 z_{10} = +44$,	$\delta'^2 z_{02} = -556$
$\delta^2 z_{11} = -38$,	$\delta'^2 z_{12} = -561$

$\phi\psi z_{00} + \phi\chi z_{01} + \theta\psi z_{10} + \theta\chi z_{11} = \cdot 2469,9167$

$-\frac{1}{6}\theta\phi\ (1+\phi)\ (\psi\delta^2 z_{00} + \chi\delta^2 z_{01}) + 1683$

$-\frac{1}{6}\theta\phi\ (1+\theta)\ (\psi\delta^2 z_{10} + \chi\delta^2 z_{11}) + 176$

$-\frac{1}{6}\psi\chi\ (4+\psi)\ (\phi\delta'^2 z_{01} + \theta\delta'^2 z_{11}) + 5230$

$+\frac{1}{6}\psi\chi\ (1+\psi)\ (\phi\delta'^2 z_{02} + \theta\delta'^2 z_{12}) - 1115$

$d_4/N = \cdot 2470,5141$

$r = \cdot 55$

$z_{00} = \cdot 279,133$,	$z_{01} = \cdot 264,313$
$z_{10} = \cdot 256,963$,	$z_{11} = \cdot 243,907$
$\delta^2 z_{00} = -319$,	$\delta'^2 z_{01} = -845$
$\delta^2 z_{01} = -391$,	$\delta'^2 z_{11} = -821$
$\delta^2 z_{10} = -46$,	$\delta'^2 z_{02} = -678$
$\delta^2 z_{11} = -137$,	$\delta'^2 z_{12} = -679$

The five several contributions to $z_{\theta,\chi}$ are:

$\cdot 2557,1206$
$+ \quad 2381$
$+ \quad 789$
$+ \quad 6110$
$- \quad 1355$
$\cdot 2557,9131$

$$\frac{d_4}{N} = \cdot 2470,5141 + \frac{189}{500}(873990)$$

$$= \cdot 2503,5509$$

$$d_4 = 250 \cdot 355$$

If we used only the hyperbolic terms:

$d_4 = 250 \cdot 288$

But $n_v' = b_4$

$= 363 - 250 \cdot 355$

$= 112 \cdot 645$ from γ

$= 112 \cdot 712$

from α

Hence $n_{st} = 182 \cdot 088 - 97 \cdot 319 - 112 \cdot 645 + 56 \cdot 437 = 28 \cdot 561$; or from β, $n_{st} = 182 \cdot 171 - 97 \cdot 349 - 112 \cdot 712 + 56 \cdot 674 = 28 \cdot 784$.

The following table shows the order of differences from the observed values:

	d_1	d_2	d_3	d_4
(i) From β and γ formulae	256·681	56·437	145·088	250·355
(ii) From α formula ...	256·651	56·674	145·171	250·288
(iii) Observed values ...	277	43	132	255
(iv) Difference (i)—(ii) ...	+ ·030	− ·237	− ·083	+ ·067
(v) Difference (i)—(iii) ...	−20·319	+13·437	+13·088	−4·645
(vi) S.D. of (i)	13·813	7·297	11·137	13·700
(vii) Ratio of (v) to (vi) ...	− 1·47	+1·84	+ 1·18	− ·34

The ratio (vii) is in no case beyond the bounds of random sampling, and since n_u contains n_u', n_v' contains n_u' and n_v contains n_u, n_u' and n_v' we should expect a high correlation between all these deviations. If we consider the actual number 27 in the chosen cell it is clearly an easy random sample from a population containing either 28·4 or 28·8 in this cell.

We will now take illustrations of the reverse process of finding r from the observed d/N.

Illustration III. The following table indicates the relation between Athletic Capacity and Intelligence in 1708 Schoolboys:

	"Intelligent" and above	"Slow Intelligent" and below	Totals
Athletic	581·25	566·75	1148
Non-athletic	209·25	350·75	560
Totals	790·5	917·5	1708

Re-arranged in standard form:

a=566·75	b=581·25	1148
c=350·75	d=209·25	560
917·5	790·5	1708

and the correlation in this form is *negative*, i.e. in the original table it is positive or the more intelligent boys are the more athletic.

We have:

$$d/N = \cdot 122{,}5117\,; \quad \tfrac{1}{2}(1-\alpha_h) = \cdot 462{,}822, \quad \tfrac{1}{2}(1-\alpha_k) = \cdot 327{,}869.$$

Hence by linear interpolation from Table XXIX of *Tables for Statisticians*,

$$h = \cdot 09333, \qquad k = \cdot 44580.$$

Our present tables show that, for d/N lying between ·115 and ·125, and h between ·0 and ·1 and k between ·4 and ·5, we must deal with the values of r, $-\cdot 20$ and $-\cdot 25$. We have first then to find the value of d/N for the above values of h and k when $r = -\cdot 20$ and $-\cdot 25$.

We need here a "single finial" formula for x (h) because we are for this variate on the border of our table. The appropriate formula is γ *bis* *, or:

$$\begin{aligned} z_{\theta,\chi} = {} & \phi\psi\, z_{0,0} + \phi\chi\, z_{0,1} + \theta\psi\, z_{1,0} + \theta\chi\, z_{1,1} \\ & - \tfrac{1}{6}\theta\phi\ \{(4+\phi)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1}) - (1+\phi)(\psi\delta^2 z_{2,0} + \chi\delta^2 z_{2,1})\} \\ & - \tfrac{1}{6}\psi\chi\,\{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0}) + (1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\}. \end{aligned}$$

In our case:

$$\theta = \cdot 9333, \quad \phi = \cdot 0667\,; \quad \chi = \cdot 4580, \quad \psi = \cdot 5420\,;$$

$$\phi\psi = \cdot 03615, \quad \phi\chi = \cdot 03055, \quad \theta\psi = \cdot 50585, \quad \theta\chi = \cdot 42745\,;$$

$$\tfrac{1}{6}\theta\phi = \cdot 01038, \quad \tfrac{1}{6}\psi\chi = \cdot 04137.$$

For $r = -\cdot 20$:

$$z_{00} = \cdot 142{,}7384, \quad z_{10} = \cdot 129{,}2840, \quad z_{01} = \cdot 126{,}0358, \quad z_{11} = \cdot 114{,}0334\,;$$

$$\delta^2 z_{10} = 4265, \quad \delta^2 z_{20} = 5421, \quad \delta^2 z_{11} = 3979, \quad \delta^2 z_{21} = 4999,$$

$$\delta'^2 z_{00} = 9853, \quad \delta'^2 z_{10} = 9221, \quad \delta'^2 z_{01} = 10914, \quad \delta'^2 z_{11} = 10163.$$

For $r = -\cdot 25$:

$$z_{00} = \cdot 135{,}2305, \quad z_{10} = \cdot 121{,}8861, \quad z_{01} = \cdot 118{,}8755, \quad z_{11} = \cdot 106{,}9947\,;$$

$$\delta^2 z_{10} = 5012, \quad \delta^2 z_{20} = 6113, \quad \delta^2 z_{11} = 4684, \quad \delta^2 z_{21} = 5644,$$

$$\delta'^2 z_{00} = 10508, \quad \delta'^2 z_{10} = 9851, \quad \delta'^2 z_{01} = 11470, \quad \delta'^2 z_{11} = 10691.$$

From these two sets of values we can write down the values of $z_{\theta,\chi}$, i.e. those of d/N, for the above formula. There results the following numbers:

$z_{\theta,\chi} = d/N =$	When $r = -\cdot 20$	When $r = -\cdot 25$
$\phi\psi\, z_{0,0} + \phi\chi\, z_{0,1} + \theta\psi\, z_{1,0} + \theta\chi\, z_{1,1}$	$= \cdot 1231{,}5228$	$= \cdot 1159{,}1120$
$-\tfrac{1}{6}\theta\phi\ \{(4+\phi)(\psi\delta^2 z_{1,0} + \chi\delta^2 z_{1,1})\}$	$- \quad 1745$	$- \quad 2052$
$+\tfrac{1}{6}\theta\phi\ \{(1+\phi)(\psi\delta^2 z_{2,0} + \chi\delta^2 z_{2,1})\}$	$+ \quad 579$	$+ \quad 653$
$-\tfrac{1}{6}\psi\chi\,\{(1+\psi)(\phi\delta'^2 z_{0,0} + \theta\delta'^2 z_{1,0})\}$	$- \quad 5909$	$- \quad 6312$
$-\tfrac{1}{6}\psi\chi\,\{(1+\chi)(\phi\delta'^2 z_{0,1} + \theta\delta'^2 z_{1,1})\}$	$- \quad 6160$	$- \quad 6480$
	$\cdot 1230{,}1993$	$\cdot 1157{,}6929$

* See *Biometrika*, Vol. XIX. p. 358, and Diagram, p. 357.

Hence by linear interpolation:

$$r = -\cdot 20 - \frac{5082}{72501} \times \cdot 05 = -\cdot 2035.$$

If we use only the first term, the hyperbolic formula, we have:

$$r = -\cdot 20 - \frac{6406}{72411} \times \cdot 05 = -\cdot 2044.$$

This is not so close to the full interpolation formula value, but would be a sufficiently close value of r for most practical purposes.

Illustration IV. The following table illustrates the influence of Wage of Father on the nature of the Mother's Employment:

Employment of Mother.

Wage of Father	Homework	Outwork	Totals
Under 22/-	144	106	250
22/- and over	168	39	207
Totals	312	145	457

Clearly d is the category 39, and the correlation in the table thus arranged is negative. We have:

$$\tfrac{1}{2}(1-\alpha_h) = \frac{145}{457} = \cdot 317{,}287\,; \quad \tfrac{1}{2}(1-\alpha_k) = \frac{207}{457} = \cdot 452{,}954.$$

Accordingly

$$d/N = \cdot 085{,}3392, \text{ and } h = \cdot 47530, \quad k = \cdot 11821.$$

$$\theta = \cdot 7530, \quad \phi = \cdot 2470\,; \quad \chi = \cdot 1821, \quad \psi = \cdot 8179\,;$$

$$\phi\psi = \cdot 20202, \quad \phi\chi = \cdot 04498, \quad \theta\psi = \cdot 61588, \quad \theta\chi = \cdot 13712\,;$$

$$\tfrac{1}{6}\theta\phi = \cdot 03100, \quad \tfrac{1}{6}\psi\chi = \cdot 02482.$$

The above value of d/N, for a value of h between ·40 and ·50, and of k between ·10 and ·20, lies between the values for $r = -\cdot 40$ and $r = -\cdot 45$:

$$r = -\cdot 40 \begin{cases} z_{0,0} = \cdot 099{,}2408, & z_{1,0} = \cdot 085{,}5431, & z_{0,1} = \cdot 087{,}0997, & z_{1,1} = \cdot 074{,}8716\,; \\ \delta^2 z_{0,0} = 11853, & \delta^2 z_{1,0} = 12327, & \delta^2 z_{0,1} = 11026, & \delta^2 z_{1,1} = 11382, \\ \delta'^2 z_{0,0} = 7394, & \delta'^2 z_{1,0} = 6895, & \delta'^2 z_{0,1} = 8263, & \delta'^2 z_{1,1} = 7616. \end{cases}$$

$$r = -\cdot 45 \begin{cases} z_{0,0} = \cdot 091{,}4776, & z_{1,0} = \cdot 078{,}2330, & z_{0,1} = \cdot 079{,}6341, & z_{1,1} = \cdot 067{,}8782\,; \\ \delta^2 z_{0,0} = 12573, & \delta^2 z_{1,0} = 12899, & \delta^2 z_{0,1} = 11675, & \delta^2 z_{1,1} = 11878, \\ \delta'^2 z_{0,0} = 8255, & \delta'^2 z_{1,0} = 7680, & \delta'^2 z_{0,1} = 9021, & \delta'^2 z_{1,1} = 8289. \end{cases}$$

Accordingly we have the following values for:

$z_{\theta,\chi}=d/N=$	When $r=-\cdot40$	When $r=-\cdot45$
$\phi\psi z_{0,0}+\phi\chi z_{0,1}+\theta\psi z_{1,0}+\theta\chi z_{1,1}$	$=\cdot0869,1705$	$=\cdot0795,5185$
$-\frac{1}{6}\theta\phi\ \{(1+\phi)(\psi\delta^2 z_{0,0}+\chi\delta^2 z_{0,1})\}$	$-\quad4524$	$-\quad4797$
$-\frac{1}{6}\theta\phi\ \{(1+\theta)(\psi\delta^2 z_{1,0}+\chi\delta^2 z_{1,1})\}$	$-\quad6605$	$-\quad6909$
$-\frac{1}{6}\chi\psi\ \{(1+\psi)(\phi\delta'^2 z_{0,0}+\theta\delta'^2 z_{1,0})\}$	$-\quad3167$	$-\quad3529$
$-\frac{1}{6}\chi\psi\ \{(1+\chi)(\phi\delta'^2 z_{0,1}+\theta\delta'^2 z_{1,1})\}$	$-\quad2281$	$-\quad2485$
	$\cdot0867,5128$	$\cdot0793,7465$

Hence by linear interpolation

$$r=-\cdot40-\frac{14121}{73766}\times\cdot05=-\cdot4096,$$

or the correlation of Mother's increasing Outwork with Father's decreasing Wage is ·4096.

Had we used only the hyperbolic formula, or the first line of the above expression for $z_{\theta,\chi}$, we should have found

$$r=-\cdot40-\frac{15778}{73652}\times\cdot05=-\cdot40-\cdot0107$$

$$=-\cdot4107, \text{ as above.}$$

Thus, in all the examples tried in this introduction, it would appear as if the hyperbolic formula were adequate for either finding a cell content, or determining the value of the coefficient of correlation.

TABLES OF THE VOLUMES OF THE NORMAL SURFACE.

k	d/N for $r = \cdot00$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2500000	·2300861	·2103702	·1910443	·1722891	·1542688	·1371266	·1209818	·1059277	0·0
0·1	·2300861	·2117584	·1936130	·1758265^{+}	·1585653	·1419804	·1262037	·1113449	·0974900	0·1
0·2	·2103702	·1936130	·1770224	·1607601	·1449780	·1298142	·1153893	·1018039	·0891361	0·2
0·3	·1910443	·1758265^{+}	·1607601	·1459917	·1316594	·1178887	·1047890	·0924515^{+}	·0809475^{+}	0·3
0·4	·1722891	·1585653	·1449780	·1316594	·1187342	·1063153	·0945017	·0833754	·0730008	0·4
0·5	·1542688	·1419804	·1298142	·1178887	·1063153	·0951954	·0846174	·0746549	·0653653	0·5
0·6	·1371266	·1262037	·1153893	·1047890	·0945017	·0846174	·0752148	·0663593	·0581020	0·6
0·7	·1209818	·1113449	·1018039	·0924515^{+}	·0833754	·0746549	·0663593	·0585464	·0512613	0·7
0·8	·1059277	·0974900	·0891361	·0809475^{+}	·0730008	·0653653	·0581020	·0512613	·0448827	0·8
0·9	·0920301	·0846993	·0774415^{+}	·0703273	·0634231	·0567895^{-}	·0504791	·0445359	·0389941	0·9
1·0	·0793276	·0730087	·0667527	·0606204	·0546692	·0489511	·0435117	·0383888	·0336120	1·0
1·1	·0678330	·0624297	·0570802	·0518365^{-}	·0467476	·0418581	·0372068	·0328263	·0287416	1·1
1·2	·0575348	·0529519	·0484144	·0439668	·0396505^{+}	·0355033	·0315582	·0278427	·0243781	1·2
1·3	·0484002	·0445449	·0407279	·0369864	·0333553	·0298666	·0265478	·0234222	·0205077	1·3
1·4	·0403783	·0371620	·0339776	·0308562	·0278270	·0249165^{-}	·0221478	·0195402	·0171087	1·4
1·5	·0334036	·0307428	·0281085^{-}	·0255263	·0230203	·0206125^{+}	·0183221	·0161649	·0141535^{-}	1·5
1·6	·0273996	·0252171	·0230563	·0209382	·0188826	·0169076	·0150289	·0132594	·0116095^{+}	1·6
1·7	·0222827	·0205078	·0187505^{-}	·0170280	·0153563	·0137501	·0122222	·0107832	·0094414	1·7
1·8	·0179652	·0165341	·0151173	·0137286	·0123808	·0110859	·0098540	·0086938	·0076120	1·8
1·9	·0143583	·0132146	·0120822	·0109723	·0098951	·0088601	·0078756	·0069484	·0060838	1·9
2·0	·0113751	·0104690	·0095719	·0086926	·0078392	·0070193	·0062393	·0055047	·0048197	2·0
2·1	·0089322	·0082207	·0075163	·0068258	·0061557	·0055118	·0048994	·0043225^{+}	·0037847	2·1
2·2	·0069517	·0063980	·0058497	·0053123	·0047908	·0042897	·0038131	·0033641	·0029455^{+}	2·2
2·3	·0053621	·0049349	·0045121	·0040976	·0036953	·0033088	·0029411	·0025948	·0022720	2·3
2·4	·0040988	·0037723	·0034490	·0031322	·0028247	·0025292	·0022482	·0019835^{+}	·0017367	2·4
2·5	·0031048	·0028575^{+}	·0026127	·0023726	·0021397	·0019159	·0017030	·0015025^{+}	·0013156	2·5
2·6	·0023306	·0021449	·0019611	·0017810	·0016061	·0014382	·0012783	·0011278	·0009875^{-}	2·6

k	d/N for $r = \cdot00$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0920301	·0793276	·0678330	·0575348	·0484002	·0403783	·0334036	·0273996	·0222827	0·0
0·1	·0846993	·0730087	·0624297	·0529519	·0445449	·0371620	·0307428	·0252171	·0205078	0·1
0·2	·0774415^{+}	·0667527	·0570802	·0484144	·0407279	·0339776	·0281085^{-}	·0230563	·0187505^{-}	0·2
0·3	·0703273	·0606204	·0518365^{-}	·0439668	·0369864	·0308562	·0255263	·0209382	·0170280	0·3
0·4	·0634231	·0546692	·0467476	·0396505^{+}	·0333553	·0278270	·0230203	·0188826	·0153563	0·4
0·5	·0567895^{-}	·0489511	·0418581	·0355033	·0298666	·0249165^{-}	·0206125^{+}	·0169076	·0137501	0·5
0·6	·0504791	·0435117	·0372068	·0315582	·0265478	·0221478	·0183221	·0150289	·0122222	0·6
0·7	·0445359	·0383888	·0328263	·0278427	·0234222	·0195402	·0161649	·0132594	·0107832	0·7
0·8	·0389941	·0336120	·0287416	·0243781	·0205077	·0171087	·0141535^{-}	·0116095^{+}	·0094414	0·8
0·9	·0338781	·0292021	·0249707	·0211797	·0178171	·0148641	·0122965^{+}	·0100864	·0082027	0·9
1·0	·0292021	·0251715^{-}	·0215241	·0182564	·0153579	·0128125^{-}	·0105993	·0086942	·0070705^{+}	1·0
1·1	·0249707	·0215241	·0184053	·0156110	·0131325^{+}	·0109559	·0090635^{-}	·0074344	·0060460	1·1
1·2	·0211797	·0182564	·0156110	·0132410	·0111388	·0092926	·0076875^{-}	·0063057	·0051281	1·2
1·3	·0178171	·0153579	·0131325^{+}	·0111388	·0093703	·0078173	·0064670	·0053046	·0043140	1·3
1·4	·0148641	·0128125^{-}	·0109559	·0092926	·0078173	·0065216	·0053951	·0044254	·0035990	1·4
1·5	·0122965^{+}	·0105993	·0090635^{-}	·0076875^{-}	·0064670	·0053951	·0044632	·0036610	·0029773	1·5
1·6	·0100864	·0086942	·0074344	·0063057	·0053046	·0044254	·0036610	·0030030	·0024422	1·6
1·7	·0082027	·0070705^{+}	·0060460	·0051281	·0043140	·0035990	·0029773	·0024422	·0019861	1·7
1·8	·0066133	·0057005^{+}	·0048745^{+}	·0041345^{-}	·0034781	·0029016	·0024004	·0019690	·0016013	1·8
1·9	·0052856	·0045560	·0038959	·0033044	·0027798	·0023191	·0019185^{-}	·0015736	·0012798	1·9
2·0	·0041874	·0036094	·0030864	·0026179	·0022022	·0018372	·0015199	·0012467	·0010139	2·0
2·1	·0032881	·0028343	·0024236	·0020557	·0017293	·0014427	·0011935^{-}	·0009790	·0007961	2·1
2·2	·0025591	·0022059	·0018862	·0015999	·0013459	·0011228	·0009288	·0007619	·0006196	2·2
2·3	·0019739	·0017014	·0014549	·0012340	·0010381	·0008660	·0007164	·0005877	·0004779	2·3
2·4	·0015088	·0013006	·0011121	·0009433	·0007935^{+}	·0006620	·0005477	·0004492	·0003653	2·4
2·5	·0011430	·0009852	·0008424	·0007145^{+}	·0006011	·0005015^{-}	·0004149	·0003403	·0002767	2·5
2·6	·0008579	·0007395^{+}	·0006324	·0005364	·0004512	·0003764	·0003114	·0002554	·0002077	2·6

k	d/N for $r = \cdot 00$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0179652	·0143583	·0113751	·0089322	·0069517	·0053621	·0040988	·0031048	·0023306	0·0
0·1	·0165341	·0132146	·0104690	·0082207	·0063980	·0049349	·0037723	·0028575$^+$	·0021449	0·1
0·2	·0151173	·0120822	·0095719	·0075163	·0058497	·0045121	·0034490	·0026127	·0019611	0·2
0·3	·0137286	·0109723	·0086926	·0068258	·0053123	·0040976	·0031322	·0023726	·0017810	0·3
0·4	·0123808	·0098951	·0078392	·0061557	·0047908	·0036953	·0028247	·0021397	·0016061	0·4
0·5	·0110859	·0088601	·0070193	·0055118	·0042897	·0033088	·0025292	·0019159	·0014382	0·5
0·6	·0098540	·0078756	·0062393	·0048994	·0038131	·0029411	·0022482	·0017030	·0012783	0·6
0·7	·0086938	·0069484	·0055047	·0043225$^+$	·0033641	·0025948	·0019835$^+$	·0015025$^+$	·0011278	0·7
0·8	·0076120	·0060838	·0048197	·0037847	·0029455$^+$	·0022720	·0017367	·0013156	·0009875$^-$	0·8
0·9	·0066133	·0052856	·0041874	·0032881	·0025591	·0019739	·0015088	·0011430	·0008579	0·9
1·0	·0057005$^+$	·0045560	·0036094	·0028343	·0022059	·0017014	·0013006	·0009852	·0007395$^+$	1·0
1·1	·0048745$^+$	·0038959	·0030864	·0024236	·0018862	·0014549	·0011121	·0008424	·0006324	1·1
1·2	·0041345$^-$	·0033044	·0026179	·0020557	·0015999	·0012340	·0009433	·0007145$^+$	·0005364	1·2
1·3	·0034781	·0027798	·0022022	·0017293	·0013459	·0010381	·0007935$^+$	·0006011	·0004512	1·3
1·4	·0029016	·0023191	·0018372	·0014427	·0011228	·0008660	·0006620	·0005015$^-$	·0003764	1·4
1·5	·0024004	·0019185$^-$	·0015199	·0011935$^-$	·0009288	·0007164	·0005477	·0004149	·0003114	1·5
1·6	·0019690	·0015736	·0012467	·0009790	·0007619	·0005877	·0004492	·0003403	·0002554	1·6
1·7	·0016013	·0012798	·0010139	·0007961	·0006196	·0004779	·0003653	·0002767	·0002077	1·7
1·8	·0012910	·0010318	·0008174	·0006419	·0004996	·0003853	·0002945$^+$	·0002231	·0001675$^-$	1·8
1·9	·0010318	·0008246	·0006533	·0005130	·0003993	·0003080	·0002354	·0001783	·0001339	1·9
2·0	·0008174	·0006533	·0005176	·0004064	·0003163	·0002440	·0001865$^-$	·0001413	·0001060	2·0
2·1	·0006419	·0005130	·0004064	·0003191	·0002484	·0001916	·0001464	·0001109	·0000833	2·1
2·2	·0004996	·0003993	·0003163	·0002484	·0001933	·0001491	·0001140	·0000863	·0000648	2·2
2·3	·0003853	·0003080	·0002440	·0001916	·0001491	·0001150$^+$	·0000879	·0000666	·0000500$^-$	2·3
2·4	·0002945$^+$	·0002354	·0001865$^-$	·0001464	·0001140	·0000879	·0000672	·0000509	·0000382	2·4
2·5	·0002231	·0001783	·0001413	·0001109	·0000863	·0000666	·0000509	·0000386	·0000289	2·5
2·6	·0001675$^-$	·0001339	·0001060	·0000833	·0000648	·0000500$^-$	·0000382	·0000289	·0000217	2·6

k	d/N for $r = -\cdot 05$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2420389	·2221648	·2025669	·1834338	·1649406	·1472439	·1304779	·1147519	·1001483	0·0
0·1	·2221648	·2038786	·1858525$^+$	·1682597	·1512608	·1349993	·1195981	·1051570	·0917509	0·1
0·2	·2025669	·1858525$^+$	·1693814	·1533116	·1377895$^+$	·1229457	·1088920	·0957188	·0834939	0·2
0·3	·1834338	·1682597	·1533116	·1387327	·1246556	·1111983	·0984617	·0865272	·0754557	0·3
0·4	·1649406	·1512608	·1377895$^+$	·1246556	·1119782	·0998633	·0884013	·0776650$^+$	·0677086	0·4
0·5	·1472439	·1349993	·1229457	·1111983	·0998633	·0890353	·0787944	·0692055$^-$	·0603163	0·5
0·6	·1304779	·1195981	·1088920	·0984617	·0884013	·0787944	·0697119	·0612108	·0533329	0·6
0·7	·1147519	·1051570	·0957188	·0865272	·0776650$^+$	·0692055$^-$	·0612108	·0537306	·0468015$^-$	0·7
0·8	·1001483	·0917509	·0834939	·0754557	·0677086	·0603163	·0533329	·0468015$^-$	·0407536	0·8
0·9	·0867220	·0794297	·0722620	·0652871	·0585674	·0521580	·0461055$^-$	·0404469	·0352093	0·9
1·0	·0745010	·0682182	·0620453	·0560408	·0502582	·0447449	·0395407	·0346772	·0301773	1·0
1·1	·0634879	·0581182	·0528445$^+$	·0477168	·0427806	·0380762	·0336373	·0294907	·0256557	1·1
1·2	·0536621	·0491100	·0446412	·0402978	·0361184	·0321368	·0283815$^+$	·0248749	·0216332	1·2
1·3	·0449829	·0411557	·0374000	·0337512	·0302417	·0268996	·0237489	·0208080	·0180903	1·3
1·4	·0373929	·0342018	·0310717	·0280320	·0251095$^+$	·0223277	·0197062	·0172603	·0150011	1·4
1·5	·0308214	·0281832	·0255964	·0230854	·0206723	·0183763	·0162135$^-$	·0141964	·0123341	1·5
1·6	·0251885$^+$	·0230258	·0209062	·0188496	·0168740	·0149951	·0132259	·0115767	·0100547	1·6
1·7	·0204082	·0186505$^+$	·0169286	·0152586	·0136551	·0121307	·0106960	·0093591	·0081259	1·7
1·8	·0163918	·0149757	·0135890	·0122446	·0109544	·0097283	·0085749	·0075006	·0065100	1·8
1·9	·0130509	·0119198	·0108128	·0097401	·0087109	·0077334	·0068143	·0059585$^+$	·0051699	1·9
2·0	·0102994	·0094041	·0085281	·0076796	·0068660	·0060935$^-$	·0053674	·0046918	·0040693	2·0
2·1	·0080561	·0073535$^+$	·0066665$^+$	·0060013	·0053638	·0047587	·0041903	·0036616	·0031747	2·1
2·2	·0062452	·0056989	·0051648	·0046480	·0041528	·0036832	·0032421	·0028320	·0024546	2·2
2·3	·0047980	·0043769	·0039655$^+$	·0035676	·0031865$^-$	·0028251	·0024860	·0021708	·0018808	2·3
2·4	·0036529	·0033313	·0030173	·0027136	·0024229	·0021474	·0018890	·0016489	·0014281	2·4
2·5	·0027560	·0025125$^+$	·0022749	·0020453	·0018256	·0016175$^-$	·0014223	·0012411	·0010745$^+$	2·5
2·6	·0020603	·0018775$^-$	·0016996	·0015276	·0013630	·0012072	·0010612	·0009257	·0008011	2·6

k	d/N for $r = -\cdot05$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0867220	·0745010	·0634879	·0536621	·0449829	·0373929	·0308214	·0251885^{+}	·0204082	0·0
0·1	·0794297	·0682182	·0581182	·0491100	·0411557	·0342018	·0281832	·0230258	·0186505^{+}	0·1
0·2	·0722620	·0620453	·0528445^{+}	·0446412	·0374000	·0310717	·0255964	·0209062	·0169286	0·2
0·3	·0652871	·0560408	·0477168	·0402978	·0337512	·0280320	·0230854	·0188496	·0152586	0·3
0·4	·0585674	·0502582	·0427806	·0361184	·0302417	·0251095^{+}	·0206723	·0168740	·0136551	0·4
0·5	·0521580	·0447449	·0380762	·0321368	·0268996	·0223277	·0183763	·0149951	·0121307	0·5
0·6	·0461055^{-}	·0395407	·0336373	·0283815^{+}	·0237489	·0197062	·0162135^{-}	·0132259	·0106960	0·6
0·7	·0404469	·0346772	·0294907	·0248749	·0208080	·0172603	·0141964	·0115767	·0093591	0·7
0·8	·0352093	·0301773	·0256557	·0216332	·0180903	·0150011	·0123341	·0100547	·0081259	0·8
0·9	·0304097	·0260554	·0221442	·0186661	·0156040	·0129350^{-}	·0106318	·0086640	·0069995^{+}	0·9
1·0	·0260554	·0223172	·0189610	·0159776	·0133520	·0110644	·0090911	·0074059	·0059810	1·0
1·1	·0221442	·0189610	·0161041	·0135656	·0113325^{+}	·0093877	·0077107	·0062792	·0050693	1·1
1·2	·0186661	·0159776	·0135656	·0114254	·0095396	·0078997	·0064863	·0052802	·0042612	1·2
1·3	·0156040	·0133520	·0113325^{+}	·0095396	·0079637	·0065924	·0054109	·0044032	·0035522	1·3
1·4	·0129350^{-}	·0110644	·0093877	·0078997	·0065924	·0054552	·0044759	·0036410	·0029362	1·4
1·5	·0106318	·0090911	·0077107	·0064863	·0054109	·0044759	·0036711	·0029852	·0024065^{-}	1·5
1·6	·0086640	·0074059	·0062792	·0052802	·0044032	·0036410	·0029852	·0024265^{+}	·0019554	1·6
1·7	·0069995^{+}	·0059810	·0050693	·0042612	·0035522	·0029362	·0024065^{-}	·0019554	·0015751	1·7
1·8	·0056057	·0047883	·0040569	·0034090	·0028407	·0023473	·0019230	·0015620	·0012578	1·8
1·9	·0044501	·0037999	·0032183	·0027033	·0022518	·0018600	·0015232	·0012368	·0009955^{+}	1·9
2·0	·0035016	·0029889	·0025305^{-}	·0021248	·0017693	·0014608	·0011959	·0009706	·0007810	2·0
2·1	·0027307	·0023301	·0019720	·0016552	·0013778	·0011371	·0009305^{+}	·0007550^{-}	·0006072	2·1
2·2	·0021106	·0018003	·0015230	·0012779	·0010633	·0008773	·0007176	·0005820	·0004679	2·2
2·3	·0016166	·0013784	·0011657	·0009777	·0008132	·0006707	·0005484	·0004446	·0003573	2·3
2·4	·0012271	·0010459	·0008842	·0007413	·0006163	·0005081	·0004153	·0003366	·0002704	2·4
2·5	·0009229	·0007864	·0006645^{+}	·0005569	·0004629	·0003814	·0003117	·0002525^{-}	·0002027	2·5
2·6	·0006879	·0005859	·0004949	·0004146	·0003445^{-}	·0002837	·0002317	·0001876	·0001506	2·6

k	d/N for $r = -\cdot05$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0163918	·0130509	·0102994	·0080561	·0062452	·0047980	·0036529	·0027560	·0020693	0·0
0·1	·0149757	·0119198	·0094041	·0073535^{+}	·0056989	·0043769	·0033313	·0025125^{+}	·0018775^{-}	0·1
0·2	·0135890	·0108128	·0085281	·0066665^{+}	·0051648	·0039655^{+}	·0030173	·0022749	·0016996	0·2
0·3	·0122446	·0097401	·0076796	·0060013	·0046480	·0035676	·0027136	·0020453	·0015276	0·3
0·4	·0109544	·0087109	·0068660	·0053638	·0041528	·0031865^{-}	·0024229	·0018256	·0013630	0·4
0·5	·0097283	·0077334	·0060935^{-}	·0047587	·0036832	·0028251	·0021474	·0016175^{-}	·0012072	0·5
0·6	·0085749	·0068143	·0053674	·0041903	·0032421	·0024860	·0018890	·0014223	·0010612	0·6
0·7	·0075006	·0059585^{+}	·0046918	·0036616	·0028320	·0021708	·0016489	·0012411	·0009257	0·7
0·8	·0065100	·0051699	·0040693	·0031747	·0024546	·0018808	·0014281	·0010745^{+}	·0008011	0·8
0·9	·0056057	·0044501	·0035016	·0027307	·0021106	·0016166	·0012271	·0009229	·0006879	0·9
1·0	·0047883	·0037999	·0029889	·0023301	·0018003	·0013784	·0010459	·0007864	·0005859	1·0
1·1	·0040569	·0032183	·0025305^{-}	·0019720	·0015230	·0011657	·0008842	·0006645^{+}	·0004949	1·1
1·2	·0034090	·0027033	·0021248	·0016552	·0012779	·0009777	·0007413	·0005569	·0004146	1·2
1·3	·0028407	·0022518	·0017693	·0013778	·0010633	·0008132	·0006163	·0004629	·0003445^{-}	1·3
1·4	·0023473	·0018600	·0014608	·0011371	·0008773	·0006707	·0005081	·0003814	·0002837	1·4
1·5	·0019230	·0015232	·0011959	·0009305^{+}	·0007176	·0005484	·0004153	·0003117	·0002317	1·5
1·6	·0015620	·0012368	·0009706	·0007550^{-}	·0005820	·0004446	·0003366	·0002525^{-}	·0001876	1·6
1·7	·0012578	·0009955^{+}	·0007810	·0006072	·0004679	·0003573	·0002704	·0002027	·0001506	1·7
1·8	·0010039	·0007943	·0006229	·0004841	·0003729	·0002846	·0002153	·0001614	·0001198	1·8
1·9	·0007943	·0006282	·0004924	·0003826	·0002946	·0002247	·0001699	·0001273	·0000945^{+}	1·9
2·0	·0006229	·0004924	·0003858	·0002996	·0002306	·0001759	·0001329	·0000996	·0000739	2·0
2·1	·0004841	·0003826	·0002996	·0002326	·0001790	·0001364	·0001031	·0000772	·0000572	2·1
2·2	·0003729	·0002946	·0002306	·0001790	·0001376	·0001049	·0000792	·0000593	·0000439	2·2
2·3	·0002846	·0002247	·0001759	·0001364	·0001049	·0000799	·0000603	·0000451	·0000334	2·3
2·4	·0002153	·0001699	·0001329	·0001031	·0000792	·0000603	·0000455^{+}	·0000340	·0000252	2·4
2·5	·0001614	·0001273	·0000996	·0000772	·0000593	·0000451	·0000340	·0000254	·0000188	2·5
2·6	·0001198	·0000945^{+}	·0000739	·0000572	·0000439	·0000334	·0000252	·0000188	·0000139	2·6

k	d/N for $r = -\cdot10$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2340579	·2142237	·1947447	·1758059	·1575766	·1402058	·1238186	·1085141	·0943638	0·0
0·1	·2142237	·1959834	·1780814	·1606873	·1439559	·1280229	·1130022	·0989832	·0860300	0·1
0·2	·1947447	·1780814	·1617382	·1458694	·1306156	·1160997	·1024242	·0896694	·0778926	0·2
0·3	·1758059	·1606873	·1458694	·1314918	·1176809	·1045474	·0921830	·0806592	·0700264	0·3
0·4	·1575766	·1439559	·1306156	·1176809	·1052651	·0934667	·0823672	·0720298	·0624986	0·4
0·5	·1402058	·1280229	·1160997	·1045474	·0934667	·0829447	·0730533	·0638480	·0553668	0·5
0·6	·1238186	·1130022	·1024242	·0921830	·0823672	·0730533	·0643044	·0561683	·0486780	0·6
0·7	·1085141	·0989832	·0896694	·0806592	·0720298	·0638480	·0561683	·0490321	·0424673	0·7
0·8	·0943638	·0860300	·0778926	·0700264	·0624986	·0553668	·0486780	·0424673	·0367586	0·8
0·9	·0814115^{+}	·0741815^{+}	·0671275^{-}	·0603139	·0537987	·0476312	·0418512	·0364888	·0315637	0·9
1·0	·0696744	·0634520	·0573857	·0515310	·0459372	·0406461	·0356916	·0310987	·0268838	1·0
1·1	·0591451	·0538329	·0486583	·0436682	·0389044	·0344021	·0301896	·0262878	·0227100	1·1
1·2	·0497937	·0452957	·0409178	·0366996	·0326759	·0288763	·0253243	·0220369	·0190251	1·2
1·3	·0415716	·0377945^{+}	·0341215^{+}	·0305853	·0272151	·0240352	·0210650^{-}	·0183184	·0158042	1·3
1·4	·0344147	·0312697	·0282138	·0252744	·0224752	·0198364	·0173736	·0150983	·0130171	1·4
1·5	·0282474	·0256509	·0231301	·0207074	·0184023	·0162311	·0142065^{+}	·0123376	·0106297	1·5
1·6	·0229861	·0208607	·0187992	·0168195^{-}	·0149376	·0131665^{+}	·0115165^{-}	·0099946	·0086051	1·6
1·7	·0185426	·0168179	·0151465^{-}	·0135429	·0120197	·0105876	·0092545^{-}	·0080260	·0069053	1·7
1·8	·0148273	·0134400	·0120967	·0108090	·0095871	·0084391	·0073715^{-}	·0063885^{+}	·0054926	1·8
1·9	·0117520	·0106459	·0095758	·0085509	·0075792	·0066672	·0058197	·0050401	·0043302	1·9
2·0	·0092319	·0083578	·0075129	·0067044	·0059386	·0052204	·0045537	·0039409	·0033834	2·0
2·1	·0071875^{+}	·0065029	·0058417	·0052097	·0046115^{-}	·0040510^{-}	·0035311	·0030538	·0026198	2·1
2·2	·0055456	·0050142	·0045014	·0040117	·0035487	·0031152	·0027135^{+}	·0023450^{-}	·0020103	2·2
2·3	·0042401	·0038313	·0034373	·0030613	·0027061	·0023739	·0020663	·0017844	·0015286	2·3
2·4	·0032124	·0029009	·0026009	·0023148	·0020448	·0017925^{+}	·0015591	·0013455^{-}	·0011517	2·4
2·5	·0024117	·0021764	·0019500^{-}	·0017343	·0015310	·0013411	·0011657	·0010052	·0008598	2·5
2·6	·0017939	·0016178	·0014486	·0012875^{-}	·0011357	·0009942	·0008635^{+}	·0007441	·0006360	2·6

k	d/N for $r = -\cdot10$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}$.	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0814115^{+}	·0696744	·0591451	·0497937	·0415716	·0344147	·0282474	·0229861	·0185426	0·0
0·1	·0741815^{+}	·0634520	·0538329	·0452957	·0377945^{+}	·0312697	·0256509	·0208607	·0168179	0·1
0·2	·0671275^{-}	·0573857	·0486583	·0409178	·0341215^{+}	·0282138	·0231301	·0187992	·0151465^{-}	0·2
0·3	·0603139	·0515310	·0436682	·0366996	·0305853	·0252744	·0207074	·0168195^{-}	·0135429	0·3
0·4	·0537987	·0459372	·0389044	·0326759	·0272151	·0224752	·0184023	·0149376	·0120197	0·4
0·5	·0476312	·0406461	·0344021	·0288763	·0240352	·0198364	·0162311	·0131665^{+}	·0105876	0·5
0·6	·0418512	·0356916	·0301896	·0253243	·0210650^{-}	·0173736	·0142065^{+}	·0115165^{-}	·0092545^{-}	0·6
0·7	·0364888	·0310987	·0262878	·0220369	·0183184	·0150983	·0123376	·0099946	·0080260	0·7
0·8	·0315637	·0268838	·0227100	·0190251	·0158042	·0130171	·0106297	·0086051	·0069053	0·8
0·9	·0270855^{-}	·0230544	·0194622	·0162932	·0135256	·0111327	·0090846	·0073490	·0058932	0·9
1·0	·0230544	·0196102	·0165435^{+}	·0138403	·0114814	·0094435^{+}	·0077007	·0062251	·0049883	1·0
1·1	·0194622	·0165435^{+}	·0139468	·0116598	·0096658	·0079445^{+}	·0064737	·0052294	·0041874	1·1
1·2	·0162932	·0138403	·0116598	·0097410	·0080693	·0066276	·0053967	·0043562	·0034856	1·2
1·3	·0135256	·0114814	·0096658	·0080693	·0066797	·0054823	·0044608	·0035981	·0028768	1·3
1·4	·0111327	·0094435^{+}	·0079445^{+}	·0066276	·0054823	·0044962	·0036557	·0029465^{+}	·0023541	1·4
1·5	·0090846	·0077007	·0064737	·0053967	·0044608	·0036557	·0029701	·0023921	·0019097	1·5
1·6	·0073490	·0062251	·0052294	·0043562	·0035981	·0029465^{+}	·0023921	·0019251	·0015357	1·6
1·7	·0058932	·0049883	·0041874	·0034856	·0028768	·0023541	·0019097	·0015357	·0012241	1·7
1·8	·0046842	·0039620	·0033234	·0027644	·0022799	·0018642	·0015111	·0012142	·0009671	1·8
1·9	·0036902	·0031190	·0026143	·0021729	·0017907	·0014631	·0011850^{+}	·0009515^{-}	·0007572	1·9
2·0	·0028812	·0024334	·0020382	·0016927	·0013939	·0011380	·0009210	·0007389	·0005876	2·0
2·1	·0022294	·0018815^{-}	·0015747	·0013068	·0010753	·0008772	·0007094	·0005687	·0004519	2·1
2·2	·0017094	·0014416	·0012056	·0009998	·0008220	·0006700	·0005414	·0004337	·0003443	2·2
2·3	·0012988	·0010945^{+}	·0009147	·0007579	·0006227	·0005071	·0004095^{-}	·0003277	·0002600	2·3
2·4	·0009779	·0008234	·0006876	·0005693	·0004673	·0003803	·0003069	·0002454	·0001945^{+}	2·4
2·5	·0007295^{+}	·0006138	·0005121	·0004237	·0003475^{+}	·0002826	·0002278	·0001820	·0001442	2·5
2·6	·0005392	·0004533	·0003779	·0003124	·0002561	·0002081	·0001676	·0001338	·0001059	2·6

k	d/N for $r = -\cdot10$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0148273	·0117520	·0092319	·0071875$^+$	·0055456	·0042401	·0032124	·0024117	·0017939	0·0
0·1	·0134400	·0106459	·0083578	·0065029	·0050142	·0038313	·0029009	·0021764	·0016178	0·1
0·2	·0120967	·0095758	·0075129	·0058417	·0045014	·0034373	·0026009	·0019500$^-$	·0014486	0·2
0·3	·0108090	·0085509	·0067044	·0052097	·0040117	·0030613	·0023148	·0017343	·0012875$^-$	0·3
0·4	·0095871	·0075792	·0059386	·0046115$^-$	·0035487	·0027061	·0020448	·0015310	·0011357	0·4
0·5	·0084391	·0066672	·0052204	·0040510	·0031152	·0023739	·0017925$^+$	·0013411	·0009942	0·5
0·6	·0073715$^-$	·0058197	·0045537	·0035311	·0027135$^+$	·0020663	·0015591	·0011657	·0008635$^+$	0·6
0·7	·0063885$^+$	·0050401	·0039409	·0030538	·0023450$^-$	·0017844	·0013455$^-$	·0010052	·0007441	0·7
0·8	·0054926	·0043302	·0033834	·0026198	·0020103	·0015286	·0011517	·0008598	·0006360	0·8
0·9	·0046842	·0036902	·0028812	·0022294	·0017094	·0012988	·0009779	·0007295$^+$	·0005392	0·9
1·0	·0039620	·0031190	·0024334	·0018815$^-$	·0014416	·0010945$^+$	·0008234	·0006138	·0004533	1·0
1·1	·0033234	·0026143	·0020382	·0015747	·0012056	·0009147	·0006876	·0005121	·0003779	1·1
1·2	·0027644	·0021729	·0016927	·0013068	·0009998	·0007579	·0005693	·0004237	·0003124	1·2
1·3	·0022799	·0017907	·0013939	·0010753	·0008220	·0006227	·0004673	·0003475$^+$	·0002561	1·3
1·4	·0018642	·0014631	·0011380	·0008772	·0006700	·0005071	·0003803	·0002826	·0002081	1·4
1·5	·0015111	·0011850$^+$	·0009210	·0007094	·0005414	·0004095$^-$	·0003069	·0002278	·0001676	1·5
1·6	·0012142	·0009515$^-$	·0007389	·0005687	·0004337	·0003277	·0002454	·0001820	·0001338	1·6
1·7	·0009671	·0007572	·0005876	·0004519	·0003443	·0002600	·0001945$^+$	·0001442	·0001059	1·7
1·8	·0007634	·0005973	·0004631	·0003558	·0002709	·0002044	·0001528	·0001132	·0000831	1·8
1·9	·0005973	·0004669	·0003618	·0002777	·0002113	·0001593	·0001190	·0000880	·0000646	1·9
2·0	·0004631	·0003618	·0002800	·0002148	·0001633	·0001230	·0000918	·0000679	·0000497	2·0
2·1	·0003558	·0002777	·0002148	·0001647	·0001251	·0000941	·0000702	·0000519	·0000379	2·1
2·2	·0002709	·0002113	·0001633	·0001251	·0000949	·0000714	·0000532	·0000393	·0000287	2·2
2·3	·0002044	·0001593	·0001230	·0000941	·0000714	·0000536	·0000399	·0000294	·0000215$^+$	2·3
2·4	·0001528	·0001190	·0000918	·0000702	·0000532	·0000399	·0000297	·0000219	·0000160	2·4
2·5	·0001132	·0000880	·0000679	·0000519	·0000393	·0000294	·0000219	·0000161	·0000118	2·5
2·6	·0000831	·0000646	·0000497	·0000379	·0000287	·0000215$^+$	·0000160	·0000118	·0000086	2·6

k	d/N for $r = -\cdot15$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2260363	·2062428	·1868846	·1681429	·1501813	·1331410	·1171378	·1022603	·0885689	0·0
0·1	·2062428	·1880530	·1702809	·1530921	·1366354	·1210385$^-$	·1064057	·0928162	·0803228	0·1
0·2	·1868846	·1702809	·1540752	·1384176	·1234424	·1092644	·0959769	·0836495$^+$	·0723285$^+$	0·2
0·3	·1681429	·1530921	·1384176	·1242545$^+$	·1107231	·0979258	·0859451	·0748422	·0646567	0·3
0·4	·1501813	·1366354	·1234424	·1107231	·0985844	·0871169	·0763930	·0664657	·0573689	0·4
0·5	·1331410	·1210385$^-$	·1092644	·0979258	·0871169	·0769171	·0673895$^-$	·0585797	·0505160	0·5
0·6	·1171378	·1064057	·0959769	·0859451	·0763930	·0673895$^-$	·0589890	·0512304	·0441371	0·6
0·7	·1022603	·0928162	·0836495$^+$	·0748422	·0664657	·0585797	·0512304	·0444507	·0382597	0·7
0·8	·0885689	·0803228	·0723285$^+$	·0646567	·0573689	·0505160	·0441371	·0382597	·0328990	0·8
0·9	·0760961	·0689532	·0620368	·0554073	·0491173	·0432098	·0377177	·0326634	·0280592	0·9
1·0	·0648481	·0587107	·0527752	·0470928	·0417081	·0366570	·0319669	·0276560	·0237339	1·0
1·1	·0548072	·0495769	·0445248	·0396943	·0351225$^+$	·0308393	·0268672	·0232209	·0199075$^+$	1·1
1·2	·0459342	·0415138	·0372495$^+$	·0331774	·0293282	·0257265$^+$	·0223907	·0193323	·0165567	1·2
1·3	·0381724	·0344680	·0308989	·0274950$^+$	·0242816	·0212786	·0185009	·0159575$^-$	·0136522	1·3
1·4	·0314513	·0283732	·0254115$^-$	·0225904	·0199306	·0174483	·0151551	·0130581	·0111599	1·4
1·5	·0256900	·0231544	·0207178	·0183999	·0162174	·0141832	·0123065$^-$	·0105925$^-$	·0090431	1·5
1·6	·0208013	·0187307	·0167436	·0148557	·0130805$^+$	·0114281	·0099056	·0085169	·0072633	1·6
1·7	·0166950$^+$	·0150190	·0134126	·0118885$^+$	·0104573	·0091268	·0079025$^+$	·0067874	·0057821	1·7
1·8	·0132807	·0119360	·0106489	·0094294	·0082857	·0072240	·0062483	·0053608	·0045618	1·8
1·9	·0104705$^-$	·0094012	·0083791	·0074121	·0065063	·0056666	·0048960	·0041961	·0035667	1·9
2·0	·0081807	·0073381	·0065338	·0057738	·0050630	·0044049	·0038018	·0032546	·0027634	2·0
2·1	·0063340	·0056760	·0050488	·0044569	·0039041	·0033930	·0029252	·0025015$^-$	·0021215$^-$	2·1
2·2	·0048595$^+$	·0043504	·0038657	·0034090	·0029830	·0025897	·0022302	·0019050$^-$	·0016138	2·2
2·3	·0036942	·0033039	·0029328	·0025836	·0022583	·0019584	·0016847	·0014374	·0012163	2·3
2·4	·0027826	·0024860	·0022045$^-$	·0019399	·0016939	·0014673	·0012608	·0010746	·0009082	2·4
2·5	·0020765$^+$	·0018533	·0016417	·0014432	·0012588	·0010892	·0009349	·0007959	·0006719	2·5
2·6	·0015352	·0013690	·0012113	·0010637	·0009267	·0008010	·0006867	·0005839	·0004924	2·6

k	d/N for $r = -\cdot15$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0760961	·0648481	·0548072	·0459342	·0381724	·0314513	·0256900	·0208013	·0166950^{+}	0·0
0·1	·0689532	·0587107	·0495769	·0415138	·0344680	·0283732	·0231544	·0187307	·0150190	0·1
0·2	·0620368	·0527752	·0445248	·0372495^{+}	·0308989	·0254115^{-}	·0207178	·0167436	·0134126	0·2
0·3	·0554073	·0470928	·0396943	·0331774	·0274950^{+}	·0225904	·0183999	·0148557	·0118885^{+}	0·3
0·4	·0491173	·0417081	·0351225^{+}	·0293282	·0242816	·0199306	·0162174	·0130805^{+}	·0104573	0·4
0·5	·0432098	·0366570	·0308393	·0257265^{+}	·0212786	·0174483	·0141832	·0114281	·0091268	0·5
0·6	·0377177	·0319669	·0268672	·0223907	·0185009	·0151551	·0123065^{-}	·0099056	·0079025^{+}	0·6
0·7	·0326634	·0276560	·0232209	·0193323	·0159575^{-}	·0130581	·0105925^{-}	·0085169	·0067874	0·7
0·8	·0280592	·0237339	·0199075^{+}	·0165567	·0136522	·0111599	·0090431	·0072633	·0057821	0·8
0·9	·0239075^{-}	·0202016	·0169272	·0140633	·0115839	·0094591	·0076566	·0061430	·0048848	0·9
1·0	·0202016	·0170525^{+}	·0142736	·0118461	·0097472	·0079506	·0064285^{+}	·0051520	·0040922	1·0
1·1	·0169272	·0142736	·0119349	·0098945^{+}	·0081325^{-}	·0066263	·0053517	·0042842	·0033991	1·1
1·2	·0140633	·0118461	·0098945^{+}	·0081940	·0067274	·0054754	·0044172	·0035321	·0027992	1·2
1·3	·0115839	·0097472	·0081325^{-}	·0067274	·0055172	·0044853	·0036144	·0028868	·0022851	1·3
1·4	·0094591	·0079506	·0066263	·0054754	·0044853	·0036422	·0029317	·0023388	·0018492	1·4
1·5	·0076566	·0064285^{+}	·0053517	·0044172	·0036144	·0029317	·0023570	·0018781	·0014832	1·5
1·6	·0061430	·0051520	·0042842	·0035321	·0028868	·0023388	·0018781	·0014948	·0011791	1·6
1·7	·0048848	·0040922	·0033991	·0027992	·0022851	·0018492	·0014832	·0011791	·0009289	1·7
1·8	·0038496	·0032213	·0026727	·0021984	·0017926	·0014489	·0011608	·0009217	·0007252	1·8
1·9	·0030065^{-}	·0025129	·0020825^{+}	·0017110	·0013935^{+}	·0011250^{+}	·0009002	·0007139	·0005611	1·9
2·0	·0023267	·0019425^{+}	·0016080	·0013195^{+}	·0010734	·0008656	·0006918	·0005479	·0004301	2·0
2·1	·0017842	·0014879	·0012302	·0010083	·0008193	·0006599	·0005267	·0004167	·0003267	2·1
2·2	·0013557	·0011292	·0009325^{+}	·0007635^{-}	·0006196	·0004984	·0003973	·0003140	·0002458	2·2
2·3	·0010206	·0008491	·0007004	·0005727	·0004642	·0003730	·0002970	·0002344	·0001833	2·3
2·4	·0007612	·0006326	·0005211	·0004256	·0003446	·0002765^{+}	·0002199	·0001733	·0001354	2·4
2·5	·0005625^{-}	·0004668	·0003842	·0003134	·0002534	·0002031	·0001613	·0001270	·0000991	2·5
2·6	·0004117	·0003413	·0002805^{+}	·0002286	·0001846	·0001478	·0001172	·0000922	·0000718	2·6

k	d/N for $r = -\cdot15$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0132807	·0104705^{-}	·0081807	·0063340	·0048595^{+}	·0036942	·0027826	·0020765^{+}	·0015352	0·0
0·1	·0119360	·0094012	·0073381	·0056760	·0043504	·0033039	·0024860	·0018533	·0013690	0·1
0·2	·0106489	·0083791	·0065338	·0050488	·0038657	·0029328	·0022045^{-}	·0016417	·0012113	0·2
0·3	·0094294	·0074121	·0057738	·0044569	·0034090	·0025836	·0019399	·0014432	·0010637	0·3
0·4	·0082857	·0065063	·0050630	·0039041	·0029830	·0022583	·0016939	·0012588	·0009267	0·4
0·5	·0072240	·0056666	·0044049	·0033930	·0025897	·0019584	·0014673	·0010892	·0008010	0·5
0·6	·0062483	·0048960	·0038018	·0029252	·0022302	·0016847	·0012608	·0009349	·0006867	0·6
0·7	·0053608	·0041961	·0032546	·0025015^{-}	·0019050^{-}	·0014374	·0010746	·0007959	·0005839	0·7
0·8	·0045618	·0035667	·0027634	·0021215^{-}	·0016138	·0012163	·0009082	·0006719	·0004924	0·8
0·9	·0038496	·0030065^{-}	·0023267	·0017842	·0013557	·0010206	·0007612	·0005625^{-}	·0004117	0·9
1·0	·0032213	·0025129	·0019425^{+}	·0014879	·0011292	·0008491	·0006326	·0004668	·0003413	1·0
1·1	·0026727	·0020825^{+}	·0016080	·0012302	·0009325^{+}	·0007004	·0005211	·0003842	·0002805^{+}	1·1
1·2	·0021984	·0017110	·0013195^{+}	·0010083	·0007635^{-}	·0005727	·0004256	·0003134	·0002286	1·2
1·3	·0017926	·0013935^{+}	·0010734	·0008193	·0006196	·0004642	·0003446	·0002534	·0001846	1·3
1·4	·0014489	·0011250^{+}	·0008656	·0006599	·0004984	·0003730	·0002765^{+}	·0002031	·0001478	1·4
1·5	·0011608	·0009002	·0006918	·0005267	·0003973	·0002970	·0002199	·0001613	·0001172	1·5
1·6	·0009217	·0007139	·0005479	·0004167	·0003140	·0002344	·0001733	·0001270	·0000922	1·6
1·7	·0007252	·0005611	·0004301	·0003267	·0002458	·0001833	·0001354	·0000991	·0000718	1·7
1·8	·0005655^{+}	·0004370	·0003346	·0002538	·0001908	·0001420	·0001048	·0000766	·0000554	1·8
1·9	·0004370	·0003372	·0002579	·0001954	·0001467	·0001091	·0000804	·0000586	·0000424	1·9
2·0	·0003346	·0002579	·0001969	·0001490	·0001117	·0000830	·0000611	·0000445^{+}	·0000321	2·0
2·1	·0002538	·0001954	·0001490	·0001126	·0000843	·0000626	·0000460	·0000335^{-}	·0000241	2·1
2·2	·0001908	·0001467	·0001117	·0000843	·0000631	·0000467	·0000343	·0000249	·0000179	2·2
2·3	·0001420	·0001091	·0000830	·0000626	·0000467	·0000346	·0000253	·0000184	·0000132	2·3
2·4	·0001048	·0000804	·0000611	·0000460	·0000343	·0000253	·0000185^{+}	·0000134	·0000097	2·4
2·5	·0000766	·0000586	·0000445^{+}	·0000335^{-}	·0000249	·0000184	·0000134	·0000097	·0000070	2·5
2·6	·0000554	·0000424	·0000321	·0000241	·0000179	·0000132	·0000097	·0000070	·0000050^{+}	2·6

k	d/N for $r = -\cdot20$									k
	$h = 0\cdot0$	$h = 0\cdot1$	$h = 0\cdot2$	$h = 0\cdot3$	$h = 0\cdot4$	$h = 0\cdot5$	$h = 0\cdot6$	$h = 0\cdot7$	$h = 0\cdot8$	
0·0	·2179529	·1982010	·1789662	·1604263	·1427384	·1260358	·1104246	·0959826	·0827586	0·0
0·1	·1982010	·1800670	·1624318	·1454567	·1292840	·1140334	·0997991	·0866492	·0746254	0·1
0·2	·1789662	·1624318	·1463745$^-$	·1309401	·1162561	·1024289	·0895418	·0776536	·0667991	0·2
0·3	·1604263	·1454567	·1309401	·1170068	·1037703	·0913243	·0797415$^-$	·0690724	·0593454	0·3
0·4	·1427384	·1292840	·1162561	·1037703	·0919265$^-$	·0808068	·0704739	·0609705$^-$	·0523195$^-$	0·4
0·5	·1260358	·1140334	·1024289	·0913243	·0808068	·0709475$^-$	·0617999	·0533996	·0457647	0·5
0·6	·1104246	·0997991	·0895418	·0797415$^-$	·0704739	·0617999	·0537647	·0463976	·0397124	0·6
0·7	·0959826	·0866492	·0776536	·0690724	·0609705$^-$	·0533996	·0463976	·0399881	·0341813	0·7
0·8	·0827586	·0746254	·0667991	·0593454	·0523195$^-$	·0457647	·0397124	·0341813	·0291784	0·8
0·9	·0707737	·0637436	·0569899	·0505683	·0445252	·0388967	·0337082	·0289744	·0246999	0·9
1·0	·0600228	·0539961	·0482160	·0427293	·0375747	·0327817	·0283709	·0243534	·0207319	1·0
1·1	·0504773	·0453540	·0404485$^+$	·0358000	·0314402	·0273932	·0236752	·0202946	·0172523	1·1
1·2	·0420888	·0377702	·0336424	·0297375$^-$	·0260814	·0226934	·0195863	·0167660	·0142324	1·2
1·3	·0347923	·0311833	·0277397	·0244877	·0214482	·0186365$^-$	·0160624	·0137300	·0116383	1·3
1·4	·0285107	·0255208	·0226730	·0199882	·0174834	·0151704	·0130565$^+$	·0111446	·0094330	1·4
1·5	·0231580	·0207028	·0183683	·0161714	·0141254	·0122394	·0105189	·0089655$^+$	·0075774	1·5
1·6	·0186436	·0166452	·0147485$^+$	·0129669	·0113105$^-$	·0097864	·0083986	·0071478	·0060321	1·6
1·7	·0148751	·0132631	·0117359	·0103039	·0089750$^+$	·0077545$^+$	·0066451	·0056471	·0047585$^-$	1·7
1·8	·0117614	·0104729	·0092543	·0081138	·0070573	·0060887	·0052099	·0044208	·0037195$^-$	1·8
1·9	·0092152	·0081945$^-$	·0072310	·0063309	·0054986	·0047370	·0040473	·0034291	·0028807	1·9
2·0	·0071542	·0063531	·0055983	·0048944	·0042448	·0036515$^+$	·0031151	·0026353	·0022104	2·0
2·1	·0055032	·0048802	·0042943	·0037490	·0032467	·0027887	·0023755$^-$	·0020065$^+$	·0016803	2·1
2·2	·0041940	·0037141	·0032635$^+$	·0028450$^-$	·0024601	·0021099	·0017945$^+$	·0015134	·0012655$^-$	2·2
2·3	·0031666	·0028003	·0024571	·0021388	·0018467	·0015814	·0013430	·0011308	·0009441	2·3
2·4	·0023685$^+$	·0020916	·0018326	·0015928	·0013732	·0011742	·0009956	·0008370	·0006977	2·4
2·5	·0017550$^-$	·0015475$^+$	·0013539	·0011751	·0010115$^+$	·0008636	·0007311	·0006137	·0005107	2·5
2·6	·0012881	·0011342	·0009908	·0008587	·0007380	·0006291	·0005318	·0004456	·0003703	2·6

k	d/N for $r = -\cdot20$									k
	$h = 0\cdot9$	$h = 1\cdot0$	$h = 1\cdot1$	$h = 1\cdot2$	$h = 1\cdot3$	$h = 1\cdot4$	$h = 1\cdot5$	$h = 1\cdot6$	$h = 1\cdot7$	
0·0	·0707737	·0600228	·0504773	·0420888	·0347923	·0285107	·0231580	·0186436	·0148751	0·0
0·1	·0637436	·0539961	·0453540	·0377702	·0311833	·0255208	·0207028	·0166452	·0132631	0·1
0·2	·0569899	·0482160	·0404485$^+$	·0336424	·0277397	·0226730	·0183683	·0147485$^+$	·0117359	0·2
0·3	·0505683	·0427293	·0358000	·0297375$^-$	·0244877	·0199882	·0161714	·0129669	·0103039	0·3
0·4	·0445252	·0375747	·0314402	·0260814	·0214482	·0174834	·0141254	·0113105$^-$	·0089750$^+$	0·4
0·5	·0388967	·0327817	·0273932	·0226934	·0186365$^-$	·0151704	·0122394	·0097864	·0077545$^+$	0·5
0·6	·0337082	·0283709	·0236752	·0195863	·0160624	·0130565$^+$	·0105189	·0083986	·0066451	0·6
0·7	·0289744	·0243534	·0202946	·0167660	·0137300	·0111446	·0089655$^+$	·0071478	·0056471	0·7
0·8	·0246999	·0207319	·0172523	·0142324	·0116383	·0094330	·0075774	·0060321	·0047585$^-$	0·8
0·9	·0208798	·0175008	·0145427	·0119797	·0097819	·0079166	·0063498	·0050472	·0039754	0·9
1·0	·0175008	·0146477	·0121542	·0099975$^+$	·0081512	·0065870	·0052753	·0041867	·0032926	1·0
1·1	·0145427	·0121542	·0100704	·0082712	·0067336	·0054331	·0043445$^+$	·0034427	·0027032	1·1
1·2	·0119797	·0099975$^+$	·0082712	·0067832	·0055138	·0044421	·0035465$^+$	·0028059	·0021997	1·2
1·3	·0097819	·0081512	·0067336	·0055138	·0044751	·0035996	·0028694	·0022666	·0017741	1·3
1·4	·0079166	·0065870	·0054331	·0044421	·0035996	·0028909	·0023008	·0018146	·0014180	1·4
1·5	·0063498	·0052753	·0043445$^+$	·0035465$^+$	·0028694	·0023008	·0018283	·0014395$^+$	·0011231	1·5
1·6	·0050472	·0041867	·0034427	·0028059	·0022666	·0018146	·0014395$^+$	·0011316	·0008814	1·6
1·7	·0039754	·0032926	·0027032	·0021997	·0017741	·0014180	·0011231	·0008814	·0006854	1·7
1·8	·0031026	·0025657	·0021031	·0017087	·0013759	·0010979	·0008681	·0006802	·0005281	1·8
1·9	·0023992	·0019809	·0016212	·0013150$^+$	·0010571	·0008422	·0006648	·0005201	·0004030	1·9
2·0	·0018381	·0015152	·0012381	·0010026	·0008047	·0006400	·0005044	·0003939	·0003048	2·0
2·1	·0013951	·0011482	·0009367	·0007573	·0006068	·0004818	·0003791	·0002955$^+$	·0002283	2·1
2·2	·0010490	·0008619	·0007020	·0005667	·0004533	·0003593	·0002822	·0002197	·0001694	2·2
2·3	·0007813	·0006410	·0005212	·0004200	·0003354	·0002654	·0002081	·0001617	·0001245$^-$	2·3
2·4	·0005765$^-$	·0004721	·0003833	·0003083	·0002458	·0001942	·0001520	·0001179	·0000906	2·4
2·5	·0004213	·0003445$^-$	·0002792	·0002242	·0001785$^-$	·0001408	·0001100	·0000852	·0000653	2·5
2·6	·0003049	·0002489	·0002014	·0001615$^-$	·0001283	·0001010	·0000788	·0000609	·0000467	2·6

k	d/N for $r = -\cdot20$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0117614	·0092152	·0071542	·0055032	·0041940	·0031666	·0023685$^+$	·0017550$^-$	·0012881	0·0
0·1	·0104729	·0081945$^-$	·0063531	·0048802	·0037141	·0028003	·0020916	·0015475$^+$	·0011342	0·1
0·2	·0092543	·0072310	·0055983	·0042943	·0032635$^+$	·0024571	·0018326	·0013539	·0009908	0·2
0·3	·0081138	·0063309	·0048944	·0037490	·0028450$^-$	·0021388	·0015928	·0011751	·0008587	0·3
0·4	·0070573	·0054986	·0042448	·0032467	·0024601	·0018467	·0013732	·0010115$^+$	·0007380	0·4
0·5	·0060887	·0047370	·0036515$^+$	·0027887	·0021099	·0015814	·0011742	·0008636	·0006291	0·5
0·6	·0052099	·0040473	·0031151	·0023755$^-$	·0017945$^+$	·0013430	·0009956	·0007311	·0005318	0·6
0·7	·0044208	·0034291	·0026353	·0020065$^+$	·0015134	·0011308	·0008370	·0006137	·0004456	0·7
0·8	·0037195$^-$	·0028807	·0022104	·0016803	·0012655$^-$	·0009441	·0006977	·0005107	·0003703	0·8
0·9	·0031026	·0023992	·0018381	·0013951	·0010490	·0007813	·0005765$^-$	·0004213	·0003049	0·9
1·0	·0025657	·0019809	·0015152	·0011482	·0008619	·0006410	·0004721	·0003445$^-$	·0002489	1·0
1·1	·0021031	·0016212	·0012381	·0009367	·0007020	·0005212	·0003833	·0002792	·0002014	1·1
1·2	·0017087	·0013150$^+$	·0010026	·0007573	·0005667	·0004200	·0003083	·0002242	·0001615$^-$	1·2
1·3	·0013759	·0010571	·0008047	·0006068	·0004533	·0003354	·0002458	·0001785$^-$	·0001283	1·3
1·4	·0010979	·0008422	·0006400	·0004818	·0003593	·0002654	·0001942	·0001408	·0001010	1·4
1·5	·0008681	·0006648	·0005044	·0003791	·0002822	·0002081	·0001520	·0001100	·0000788	1·5
1·6	·0006802	·0005201	·0003939	·0002955$^+$	·0002197	·0001617	·0001179	·0000852	·0000609	1·6
1·7	·0005281	·0004030	·0003048	·0002283	·0001694	·0001245$^-$	·0000906	·0000653	·0000467	1·7
1·8	·0004061	·0003095$^-$	·0002336	·0001747	·0001294	·0000949	·0000690	·0000496	·0000354	1·8
1·9	·0003095$^-$	·0002354	·0001774	·0001324	·0000979	·0000717	·0000520	·0000374	·0000266	1·9
2·0	·0002336	·0001774	·0001335$^-$	·0000994	·0000734	·0000537	·0000389	·0000279	·0000198	2·0
2·1	·0001747	·0001324	·0000994	·0000740	·0000545$^+$	·0000398	·0000288	·0000206	·0000146	2·1
2·2	·0001294	·0000979	·0000734	·0000545$^+$	·0000401	·0000292	·0000211	·0000151	·0000107	2·2
2·3	·0000949	·0000717	·0000537	·0000398	·0000292	·0000212	·0000153	·0000109	·0000077	2·3
2·4	·0000690	·0000520	·0000389	·0000288	·0000211	·0000153	·0000110	·0000078	·0000055$^+$	2·4
2·5	·0000496	·0000374	·0000279	·0000206	·0000151	·0000109	·0000078	·0000056	·0000039	2·5
2·6	·0000354	·0000266	·0000198	·0000146	·0000107	·0000077	·0000055$^+$	·0000039	·0000028	2·6

k	d/N for $r = -\cdot25$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2097847	·1900757	·1709684	·1526363	·1352305$^-$	·1188755$^+$	·1036675$^-$	·0896727	·0769282	0·0
0·1	·1900757	·1720036	·1545135$^+$	·1377626	·1218861	·1069947	·0931724	·0804759	·0689346	0·1
0·2	·1709684	·1545135$^+$	·1386173	·1234203	·1090429	·0955823	·0831112	·0716771	·0613028	0·2
0·3	·1526363	·1377626	·1234203	·1097345$^-$	·0968110	·0847343	·0735666	·0633471	·0540926	0·3
0·4	·1352305$^-$	·1218861	·1090429	·0968110	·0852824	·0745302	·0646066	·0555433	·0473520	0·4
0·5	·1188755$^+$	·1069947	·0955823	·0847343	·0745302	·0650321	·0562833	·0483089	·0411162	0·5
0·6	·1036675$^-$	·0931724	·0831112	·0735666	·0646066	·0562833	·0486322	·0416726	·0354080	0·6
0·7	·0896727	·0804759	·0716771	·0633471	·0555433	·0483089	·0416726	·0356484	·0302373	0·7
0·8	·0769282	·0689346	·0613028	·0540926	·0473520	·0411162	·0354080	·0302373	·0256025$^+$	0·8
0·9	·0654427	·0585527	·0519882	·0457995$^-$	·0400260	·0346964	·0298281	·0254276	·0214917	0·9
1·0	·0551995$^+$	·0493107	·0437121	·0384453	·0335425$^-$	·0290263	·0249099	·0211972	·0178837	1·0
1·1	·0461592	·0411692	·0364354	·0319918	·0278643	·0240706	·0206204	·0175153	·0147502	1·1
1·2	·0382634	·0340717	·0301040	·0263877	·0229434	·0197847	·0169182	·0143444	·0120574	1·2
1·3	·0314390	·0279489	·0246526	·0215721	·0187233	·0161167	·0137565$^+$	·0116420	·0097675$^-$	1·3
1·4	·0256019	·0227218	·0200078	·0174771	·0151421	·0130103	·0110846	·0093632	·0078406	1·4
1·5	·0206614	·0183060	·0160915$^-$	·0140313	·0121347	·0104072	·0088502	·0074616	·0062362	1·5
1·6	·0165231	·0146144	·0128239	·0111620	·0096357	·0082486	·0070012	·0058914	·0049143	1·6
1·7	·0130929	·0115604	·0101260	·0087978	·0075807	·0064772	·0054873	·0046085$^+$	·0038365$^+$	1·7
1·8	·0102794	·0090602	·0079218	·0068701	·0059086	·0050390	·0042606	·0035713	·0029672	1·8
1·9	·0079956	·0070347	·0061396	·0053147	·0045623	·0038833	·0032771	·0027414	·0022732	1·9
2·0	·0061611	·0054110	·0047138	·0040729	·0034896	·0029645$^+$	·0024968$^-$	·0020845$^+$	·0017250$^+$	2·0
2·1	·0047029	·0041228	·0035850$^+$	·0030916	·0026438	·0022416	·0018842	·0015700	·0012965$^+$	2·1
2·2	·0035560	·0031116	·0027006	·0023245$^+$	·0019840	·0016789	·0014084	·0011711	·0009652	2·2
2·3	·0026632	·0023261	·0020151	·0017311	·0014746	·0012453	·0010426	·0008652	·0007116	2·3
2·4	·0019755$^+$	·0017222	·0014891	·0012768	·0010855$^-$	·0009149	·0007644	·0006330	·0005195$^+$	2·4
2·5	·0014514	·0012629	·0010898	·0009326	·0007913	·0006656	·0005550$^-$	·0004586	·0003756	2·5
2·6	·0010560	·0009171	·0007899	·0006746	·0005713	·0004795$^+$	·0003990	·0003291	·0002690	2·6

k	d/N for $r = -\cdot25$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0654427	·0551995^{+}	·0461592	·0382634	·0314390	·0256019	·0206614	·0165231	·0130929	0·0
0·1	·0585527	·0493107	·0411692	·0340717	·0279489	·0227218	·0183060	·0146144	·0115604	0·1
0·2	·0519882	·0437121	·0364354	·0301040	·0246526	·0200078	·0160915^{-}	·0128239	·0101260	0·2
0·3	·0457995^{-}	·0384453	·0319918	·0263877	·0215721	·0174771	·0140313	·0111620	·0087978	0·3
0·4	·0400260	·0335425^{-}	·0278643	·0229434	·0187233	·0151421	·0121347	·0096357	·0075807	0·4
0·5	·0346964	·0290263	·0240706	·0197847	·0161167	·0130103	·0104072	·0082486	·0064772	0·5
0·6	·0298281	·0249099	·0206204	·0169182	·0137565^{+}	·0110846	·0088502	·0070012	·0054873	0·6
0·7	·0254276	·0211972	·0175153	·0143444	·0116420	·0093632	·0074616	·0058914	·0046085^{+}	0·7
0·8	·0214917	·0178837	·0147502	·0120574	·0097675^{-}	·0078406	·0062362	·0049143	·0038365^{+}	0·8
0·9	·0180082	·0149575^{+}	·0123138	·0100468	·0081232	·0065081	·0051662	·0040631	·0031658	0·9
1·0	·0149575^{+}	·0124005^{-}	·0101894	·0082976	·0066959	·0053541	·0042418	·0033294	·0025889	1·0
1·1	·0123138	·0101894	·0083566	·0067919	·0054701	·0043653	·0034515^{-}	·0027036	·0020980	1·1
1·2	·0100468	·0082976	·0067919	·0055093	·0044284	·0035269	·0027830	·0021755^{+}	·0016847	1·2
1·3	·0081232	·0066959	·0054701	·0044284	·0035524	·0028235^{-}	·0022234	·0017345^{-}	·0013404	1·3
1·4	·0065081	·0053541	·0043653	·0035269	·0028235^{-}	·0022396	·0017599	·0013701	·0010566	1·4
1·5	·0051662	·0042418	·0034515^{-}	·0027830	·0022234	·0017599	·0013801	·0010722	·0008251	1·5
1·6	·0040631	·0033294	·0027036	·0021755^{+}	·0017345^{-}	·0013701	·0010722	·0008312	·0006383	1·6
1·7	·0031658	·0025889	·0020980	·0016847	·0013404	·0010566	·0008251	·0006383	·0004891	1·7
1·8	·0024435^{-}	·0019942	·0016127	·0012923	·0010261	·0008071	·0006289	·0004855^{-}	·0003712	1·8
1·9	·0018682	·0015215^{+}	·0012279	·0009819	·0007780	·0006106	·0004748	·0003657	·0002790	1·9
2·0	·0014148	·0011499	·0009261	·0007390	·0005842	·0004576	·0003550^{+}	·0002729	·0002077	2·0
2·1	·0010612	·0008607	·0006917	·0005508	·0004345^{+}	·0003396	·0002629	·0002016	·0001532	2·1
2·2	·0007883	·0006380	·0005117	·0004066	·0003201	·0002496	·0001928	·0001475^{+}	·0001118	2·2
2·3	·0005800	·0004684	·0003749	·0002972	·0002335^{-}	·0001817	·0001400	·0001069	·0000809	2·3
2·4	·0004226	·0003406	·0002720	·0002152	·0001687	·0001309	·0001007	·0000767	·0000579	2·4
2·5	·0003049	·0002452	·0001954	·0001542	·0001206	·0000935^{-}	·0000717	·0000545^{+}	·0000410	2·5
2·6	·0002178	·0001748	·0001390	·0001095^{-}	·0000854	·0000660	·0000506	·0000383	·0000288	2·6

k	d/N for $r = -\cdot25$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0102794	·0079956	·0061511	·0047029	·0035560	·0026632	·0019755^{+}	·0014514	·0010560	0·0
0·1	·0090602	·0070347	·0054110	·0041228	·0031116	·0023261	·0017222	·0012629	·0009171	0·1
0·2	·0079218	·0061396	·0047138	·0035850^{+}	·0027006	·0020151	·0014891	·0010898	·0007899	0·2
0·3	·0068701	·0053147	·0040729	·0030916	·0023245^{+}	·0017311	·0012768	·0009326	·0006746	0·3
0·4	·0059086	·0045623	·0034896	·0026438	·0019840	·0014746	·0010855^{-}	·0007913	·0005713	0·4
0·5	·0050390	·0038833	·0029645^{+}	·0022416	·0016789	·0012453	·0009149	·0006656	·0004795^{+}	0·5
0·6	·0042606	·0032771	·0024968	·0018842	·0014084	·0010426	·0007644	·0005550^{-}	·0003990	0·6
0·7	·0035713	·0027414	·0020845^{+}	·0015700	·0011711	·0008652	·0006330	·0004586	·0003291	0·7
0·8	·0029672	·0022732	·0017250^{+}	·0012965^{+}	·0009652	·0007116	·0005195^{+}	·0003756	·0002690	0·8
0·9	·0024435^{-}	·0018682	·0014148	·0010612	·0007883	·0005800	·0004226	·0003049	·0002178	0·9
1·0	·0019942	·0015215^{+}	·0011499	·0008607	·0006380	·0004684	·0003406	·0002452	·0001748	1·0
1·1	·0016127	·0012279	·0009261	·0006917	·0005117	·0003749	·0002720	·0001954	·0001390	1·1
1·2	·0012923	·0009819	·0007390	·0005508	·0004066	·0002972	·0002152	·0001542	·0001095^{-}	1·2
1·3	·0010261	·0007780	·0005842	·0004345^{+}	·0003201	·0002335^{-}	·0001687	·0001206	·0000854	1·3
1·4	·0008071	·0006106	·0004576	·0003396	·0002496	·0001817	·0001309	·0000935^{-}	·0000660	1·4
1·5	·0006289	·0004748	·0003550^{+}	·0002629	·0001928	·0001400	·0001007	·0000717	·0000506	1·5
1·6	·0004855^{-}	·0003657	·0002729	·0002016	·0001475^{+}	·0001069	·0000767	·0000545^{+}	·0000383	1·6
1·7	·0003712	·0002790	·0002077	·0001532	·0001118	·0000809	·0000579	·0000410	·0000288	1·7
1·8	·0002811	·0002108	·0001566	·0001152	·0000839	·0000606	·0000433	·0000306	·0000214	1·8
1·9	·0002108	·0001578	·0001170	·0000859	·0000624	·0000449	·0000320	·0000226	·0000158	1·9
2·0	·0001566	·0001170	·0000865^{-}	·0000633	·0000459	·0000330	·0000235^{-}	·0000165^{+}	·0000115^{+}	2·0
2·1	·0001152	·0000859	·0000633	·0000463	·0000335^{-}	·0000240	·0000170	·0000120	·0000083	2·1
2·2	·0000839	·0000624	·0000459	·0000335^{-}	·0000242	·0000173	·0000122	·0000086	·0000060	2·2
2·3	·0000606	·0000449	·0000330	·0000240	·0000173	·0000123	·0000087	·0000061	·0000042	2·3
2·4	·0000433	·0000320	·0000235^{-}	·0000170	·0000122	·0000087	·0000061	·0000043	·0000030	2·4
2·5	·0000306	·0000226	·0000165^{+}	·0000120	·0000086	·0000061	·0000043	·0000030	·0000021	2·5
2·6	·0000214	·0000158	·0000115^{+}	·0000083	·0000060	·0000042	·0000030	·0000021	·0000014	2·6

k	d/N for $r = -\cdot30$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·2015067	·1818424	·1628676	·1447517	·1276390	·1116448	·0968546	·0833226	·0710731	0·0
0·1	·1818424	·1638633	·1465053	·1299903	·1144249	·0999093	·0865163	·0742903	·0632482	0·1
0·2	·1628676	·1465053	·1307846	·1158412	·1017888	·0887140	·0766782	·0657166	·0558396	0·2
0·3	·1447517	·1299903	·1158412	·1024233	·0898340	·0781481	·0674161	·0576653	·0489003	0·3
0·4	·1276390	·1144249	·1017888	·0898340	·0786444	·0682825^{+}	·0587896	·0501855^{+}	·0424703	0·4
0·5	·1116448	·0999093	·0887140	·0781481	·0682825^{+}	·0591692	·0508408	·0433110	·0365760	0·5
0·6	·0968546	·0865163	·0766782	·0674161	·0587896	·0508408	·0435950^{-}	·0370605^{+}	·0312307	0·6
0·7	·0833226	·0742903	·0657166	·0576653	·0501855^{+}	·0433110	·0370605^{+}	·0314383	·0264353	0·7
0·8	·0710731	·0632482	·0558396	·0489003	·0424703	·0365760	·0312307	·0264353	·0221793	0·8
0·9	·0601022	·0533813	·0470347	·0411056	·0356261	·0306163	·0260852	·0220311	·0184427	0·9
1·0	·0503807	·0446585^{-}	·0392691	·0342476	·0296193	·0253991	·0215924	·0181957	·0151972	1·0
1·1	·0418580	·0370290	·0324931	·0282782	·0244039	·0208808	·0177116	·0148913	·0124087	1·1
1·2	·0344655^{-}	·0304268	·0266436	·0231378	·0199240	·0170096	·0143952	·0120751	·0100384	1·2
1·3	·0281215^{+}	·0247746	·0216479	·0187585^{+}	·0161171	·0137285^{-}	·0115916	·0097007	·0080453	1·3
1·4	·0227354	·0199871	·0174269	·0150675^{+}	·0129167	·0109772	·0092470	·0077202	·0063874	1·4
1·5	·0182110	·0159753	·0138984	·0119899	·0102550^{-}	·0086949	·0073071	·0060861	·0050232	1·5
1·6	·0144511	·0126494	·0109805^{+}	·0094512	·0080649	·0068220	·0057195^{-}	·0047522	·0039127	1·6
1·7	·0113598	·0099216	·0085931	·0073794	·0062823	·0053015^{-}	·0044340	·0036751	·0030183	1·7
1·8	·0088452	·0077081	·0066608	·0057067	·0048469	·0040804	·0034044	·0028148	·0023060	1·8
1·9	·0068216	·0059312	·0051136	·0043708	·0037034	·0031102	·0025886	·0021349	·0017446	1·9
2·0	·0052104	·0045200	·0038878	·0033153	·0028023	·0023476	·0019491	·0016035^{-}	·0013069	2·0
2·1	·0039414	·0034112	·0029272	·0024902	·0020997	·0017547	·0014532	·0011925^{-}	·0009695^{+}	2·1
2·2	·0029525^{-}	·0025494	·0021825^{-}	·0018521	·0015579	·0012987	·0010728	·0008781	·0007121	2·2
2·3	·0021901	·0018866	·0016113	·0013641	·0011445^{+}	·0009517	·0007842	·0006402	·0005178	2·3
2·4	·0016086	·0013825^{-}	·0011778	·0009946	·0008325^{-}	·0006905^{-}	·0005675^{-}	·0004621	·0003728	2·4
2·5	·0011699	·0010030	·0008524	·0007181	·0005995^{+}	·0004960	·0004066	·0003302	·0002657	2·5
2·6	·0008424	·0007205^{-}	·0006108	·0005133	·0004274	·0003527	·0002884	·0002336	·0001875^{-}	2·6

k	d/N for $r = -\cdot30$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0601022	·0503807	·0418580	·0344655^{-}	·0281215^{+}	·0227354	·0182110	·0144511	·0113598	0·0
0·1	·0533813	·0446585^{-}	·0370290	·0304268	·0247746	·0199871	·0159753	·0126494	·0099216	0·1
0·2	·0470347	·0392691	·0324931	·0266436	·0216479	·0174269	·0138984	·0109805^{+}	·0085931	0·2
0·3	·0411056	·0342476	·0282782	·0231378	·0187585^{+}	·0150675^{+}	·0119899	·0094512	·0073794	0·3
0·4	·0356261	·0296193	·0244039	·0199240	·0161171	·0129167	·0102550^{-}	·0080649	·0062823	0·4
0·5	·0306163	·0253991	·0208808	·0170096	·0137285^{-}	·0109772	·0086949	·0068220	·0053015^{-}	0·5
0·6	·0260852	·0215924	·0177116	·0143952	·0115916	·0092470	·0073071	·0057195^{-}	·0044340	0·6
0·7	·0220311	·0181957	·0148913	·0120751	·0097007	·0077202	·0060861	·0047522	·0036751	0·7
0·8	·0184427	·0151972	·0124087	·0100384	·0080453	·0063874	·0050232	·0039127	·0030183	0·8
0·9	·0153006	·0125788	·0102467	·0082697	·0066118	·0052366	·0041080	·0031919	·0024562	0·9
1·0	·0125788	·0103171	·0083842	·0067503	·0053839	·0042535^{+}	·0033285^{+}	·0025797	·0019801	1·0
1·1	·0102467	·0083842	·0067970	·0054591	·0043433	·0034229	·0026718	·0020655^{-}	·0015813	1·1
1·2	·0082697	·0067503	·0054591	·0043737	·0034711	·0027286	·0021241	·0016381	·0012509	1·2
1·3	·0066118	·0053839	·0043433	·0034711	·0027478	·0021546	·0016732	·0012869	·0009801	1·3
1·4	·0052366	·0042535^{+}	·0034229	·0027286	·0021546	·0016851	·0013052	·0010012	·0007605^{+}	1·4
1·5	·0041080	·0033285^{+}	·0026718	·0021241	·0016732	·0013052	·0010084	·0007714	·0005845^{-}	1·5
1·6	·0031919	·0025797	·0020655^{-}	·0016381	·0012869	·0010012	·0007714	·0005886	·0004448	1·6
1·7	·0024562	·0019801	·0015813	·0012509	·0009801	·0007605^{+}	·0005845^{-}	·0004448	·0003352	1·7
1·8	·0018717	·0015051	·0011989	·0009459	·0007392	·0005721	·0004384	·0003328	·0002501	1·8
1·9	·0014125^{-}	·0011329	·0009000	·0007083	·0005520	·0004261	·0003257	·0002465^{-}	·0001847	1·9
2·0	·0010554	·0008443	·0006690	·0005251	·0004081	·0003142	·0002395^{-}	·0001808	·0001351	2·0
2·1	·0007808	·0006231	·0004924	·0003854	·0002988	·0002294	·0001744	·0001313	·0000978	2·1
2·2	·0005720	·0004552	·0003588	·0002801	·0002166	·0001658	·0001257	·0000944	·0000701	2·2
2·3	·0004149	·0003293	·0002588	·0002015^{+}	·0001554	·0001186	·0000897	·0000671	·0000498	2·3
2·4	·0002979	·0002358	·0001849	·0001435^{+}	·0001104	·0000840	·0000634	·0000473	·0000350^{-}	2·4
2·5	·0002118	·0001672	·0001307	·0001012	·0000776	·0000589	·0000443	·0000330	·0000243	2·5
2·6	·0001490	·0001173	·0000915^{-}	·0000706	·0000540	·0000409	·0000307	·0000228	·0000167	2·6

k	d/N for $r = -\cdot30$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0088452	·0068216	·0052104	·0039414	·0029525$^-$	·0021901	·0016086	·0011699	·0008424	0·0
0·1	·0077081	·0059312	·0045200	·0034112	·0025494	·0018866	·0013825$^-$	·0010030	·0007205$^-$	0·1
0·2	·0066608	·0051136	·0038878	·0029272	·0021825$^-$	·0016113	·0011778	·0008524	·0006108	0·2
0·3	·0057067	·0043708	·0033153	·0024902	·0018521	·0013641	·0009946	·0007181	·0005133	0·3
0·4	·0048469	·0037034	·0028023	·0020997	·0015579	·0011445$^+$	·0008325$^-$	·0005995$^+$	·0004274	0·4
0·5	·0040804	·0031102	·0023476	·0017547	·0012987	·0009517	·0006905$^-$	·0004960	·0003527	0·5
0·6	·0034044	·0025886	·0019491	·0014532	·0010728	·0007842	·0005675$^-$	·0004066	·0002884	0·6
0·7	·0028148	·0021349	·0016035$^-$	·0011925$^-$	·0008781	·0006402	·0004621	·0003302	·0002336	0·7
0·8	·0023060	·0017446	·0013069	·0009695$^+$	·0007121	·0005178	·0003728	·0002657	·0001875$^-$	0·8
0·9	·0018717	·0014125$^-$	·0010554	·0007808	·0005720	·0004149	·0002979	·0002118	·0001490	0·9
1·0	·0015051	·0011329	·0008443	·0006231	·0004552	·0003293	·0002358	·0001672	·0001173	1·0
1·1	·0011989	·0009000	·0006690	·0004924	·0003588	·0002588	·0001849	·0001307	·0000915$^-$	1·1
1·2	·0009459	·0007083	·0005251	·0003854	·0002801	·0002015$^+$	·0001435$^+$	·0001012	·0000706	1·2
1·3	·0007392	·0005520	·0004081	·0002988	·0002166	·0001554	·0001104	·0000776	·0000540	1·3
1·4	·0005721	·0004261	·0003142	·0002294	·0001658	·0001186	·0000840	·0000589	·0000409	1·4
1·5	·0004384	·0003257	·0002395$^-$	·0001744	·0001257	·0000897	·0000634	·0000443	·0000307	1·5
1·6	·0003328	·0002465$^-$	·0001808	·0001313	·0000944	·0000671	·0000473	·0000330	·0000228	1·6
1·7	·0002501	·0001847	·0001351	·0000978	·0000701	·0000498	·0000350$^-$	·0000243	·0000167	1·7
1·8	·0001861	·0001371	·0001000	·0000722	·0000516	·0000365$^+$	·0000256	·0000177	·0000122	1·8
1·9	·0001371	·0001007	·0000733	·0000527	·0000376	·0000266	·0000185$^+$	·0000128	·0000088	1·9
2·0	·0001000	·0000733	·0000531	·0000382	·0000271	·0000191	·0000133	·0000092	·0000063	2·0
2·1	·0000722	·0000527	·0000382	·0000273	·0000194	·0000136	·0000094	·0000065$^-$	·0000044	2·1
2·2	·0000516	·0000376	·0000271	·0000194	·0000137	·0000096	·0000066	·0000045$^+$	·0000031	2·2
2·3	·0000365$^+$	·0000265	·0000191	·0000136	·0000096	·0000067	·0000046	·0000032	·0000021	2·3
2·4	·0000256	·0000185$^+$	·0000133	·0000094	·0000066	·0000046	·0000032	·0000022	·0000015$^-$	2·4
2·5	·0000177	·0000128	·0000092	·0000065$^-$	·0000045$^+$	·0000032	·0000022	·0000015$^-$	·0000010	2·5
2·6	·0000122	·0000088	·0000063	·0000044	·0000031	·0000021	·0000015$^-$	·0000010	·00000066	2·6

k	d/N for $r = -\cdot35$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·1930908	·1734734	·1546379	·1367490	·1199435$^-$	·1043271	·0899735$^-$	·0769240	·0651894	0·0
0·1	·1734734	·1555469	·1383798	·1221186	·1068829	·0927635$^+$	·0798213	·0680873	·0575650$^+$	0·1
0·2	·1546379	·1383798	·1228525$^-$	·1081845$^-$	·0944794	·0818138	·0702367	·0597702	·0504113	0·2
0·3	·1367490	·1221186	·1081845$^-$	·0950584	·0828288	·0715590	·0612874	·0520282	·0437730	0·3
0·4	·1199435$^-$	·1068829	·0944794	·0828288	·0720054	·0620606	·0530233	·0449010	·0376812	0·4
0·5	·1043271	·0927635$^+$	·0818138	·0715590	·0620606	·0533593	·0454760	·0384122	·0321524	0·5
0·6	·0899735$^-$	·0798213	·0702367	·0612874	·0530233	·0454760	·0386590	·0325697	·0271903	0·6
0·7	·0769240	·0680873	·0597702	·0520282	·0449010	·0384122	·0325697	·0273672	·0227858	0·7
0·8	·0651894	·0575650$^+$	·0504113	·0437730	·0376812	·0321524	·0271903	·0227858	·0189196	0·8
0·9	·0547521	·0482324	·0421344	·0364938	·0313341	·0266664	·0224905$^-$	·0187959	·0155635$^-$	0·9
1·0	·0455699	·0400453	·0348946	·0301457	·0258156	·0219113	·0184297	·0153596	·0126824	1·0
1·1	·0375802	·0329419	·0286315$^+$	·0246704	·0210705$^-$	·0178352	·0149599	·0124328	·0102364	1·1
1·2	·0307040	·0268460	·0232727	·0199998	·0170352	·0143798	·0120277	·0099675$^+$	·0081830	1·2
1·3	·0248509	·0216723	·0187380	·0160594	·0136414	·0114829	·0095774	·0079140	·0064781	1·3
1·4	·0199230	·0173292	·0149429	·0127718	·0108186	·0090810	·0075523	·0062224	·0050784	1·4
1·5	·0158197	·0137235$^+$	·0118016	·0100765$^-$	·0084967	·0071115$^+$	·0058972	·0048444	·0039419	1·5
1·6	·0124404	·0107629	·0092301	·0078451	·0066078	·0055145$^+$	·0045595$^-$	·0037343	·0030295$^-$	1·6
1·7	·0096879	·0083586	·0071482	·0060584	·0050881	·0042339	·0034902	·0028500$^-$	·0023049	1·7
1·8	·0074704	·0064276	·0054813	·0046323	·0038791	·0032183	·0026450$^+$	·0021533	·0017364	1·8
1·9	·0057037	·0048938	·0041614	·0035067	·0029278	·0024218	·0019844	·0016105$^-$	·0012945$^-$	1·9
2·0	·0043116	·0036888	·0031278	·0026279	·0021876	·0018040	·0014737	·0011923	·0009554	2·0
2·1	·0032267	·0027527	·0023272	·0019495$^+$	·0016180	·0013303	·0010833	·0008735$^-$	·0006979	2·1
2·2	·0023905$^+$	·0020334	·0017141	·0014316	·0011846	·0009709	·0007882	·0006337	·0005046	2·2
2·3	·0017532	·0014870	·0012497	·0010406	·0008584	·0007014	·0005676	·0004549	·0003611	2·3
2·4	·0012727	·0010762	·0009018	·0007486	·0006156	·0005015$^-$	·0004046	·0003232	·0002557	2·4
2·5	·0009144	·0007710	·0006441	·0005331	·0004370	·0003548	·0002854	·0002272	·0001792	2·5
2·6	·0006503	·0005467	·0004553	·0003756	·0003070	·0002485$^-$	·0001992	·0001581	·0001243	2·6

k	d/N for $r = -\cdot35$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0547521	·0455699	·0375802	·0307040	·0248509	·0199230	·0158197	·0124404	·0096879	0·0
0·1	·0482324	·0400453	·0329419	·0268460	·0216723	·0173292	·0137235^{+}	·0107629	·0083586	0·1
0·2	·0421344	·0348946	·0286315^{+}	·0232727	·0187380	·0149429	·0118016	·0092301	·0071482	0·2
0·3	·0364938	·0301457	·0246704	·0199998	·0160594	·0127718	·0100765^{-}	·0078451	·0060584	0·3
0·4	·0313341	·0258156	·0210705^{-}	·0170352	·0136414	·0108186	·0084967	·0066078	·0050881	0·4
0·5	·0266664	·0219113	·0178352	·0143798	·0114829	·0090810	·0071115^{+}	·0055145^{+}	·0042339	0·5
0·6	·0224905^{-}	·0184297	·0149599	·0120277	·0095774	·0075523	·0058972	·0045595^{-}	·0034902	0·6
0·7	·0187959	·0153596	·0124328	·0099675^{+}	·0079140	·0062224	·0048444	·0037343	·0028500^{-}	0·7
0·8	·0155635^{-}	·0126824	·0102364	·0081830	·0064781	·0050784	·0039419	·0030295^{-}	·0023049	0·8
0·9	·0127667	·0103736	·0083488	·0066544	·0052524	·0041052	·0031768	·0024340	·0018462	0·9
1·0	·0103736	·0084048	·0067444	·0053597	·0042178	·0032866	·0025356	·0019367	·0014644	1·0
1·1	·0083488	·0067444	·0053960	·0042753	·0033542	·0026056	·0020040	·0015259	·0011502	1·1
1·2	·0066544	·0053597	·0042753	·0033771	·0026414	·0020455^{+}	·0015683	·0011904	·0008944	1·2
1·3	·0052524	·0042178	·0033542	·0026414	·0020596	·0015900	·0012152	·0009194	·0006886	1·3
1·4	·0041052	·0032866	·0026056	·0020455^{+}	·0015900	·0012236	·0009322	·0007031	·0005249	1·4
1·5	·0031768	·0025356	·0020040	·0015683	·0012152	·0009322	·0007079	·0005322	·0003960	1·5
1·6	·0024340	·0019367	·0015259	·0011904	·0009194	·0007031	·0005322	·0003988	·0002958	1·6
1·7	·0018462	·0014644	·0011502	·0008944	·0006886	·0005249	·0003960	·0002958	·0002186	1·7
1·8	·0013863	·0010961	·0008582	·0006652	·0005105^{+}	·0003878	·0002916	·0002171	·0001600	1·8
1·9	·0010304	·0008121	·0006338	·0004897	·0003746	·0002836	·0002126	·0001577	·0001158	1·9
2·0	·0007581	·0005956	·0004633	·0003568	·0002720	·0002053	·0001534	·0001134	·0000830	2·0
2·1	·0005520	·0004323	·0003352	·0002573	·0001955^{+}	·0001471	·0001095^{+}	·0000807	·0000589	2·1
2·2	·0003978	·0003105^{+}	·0002400	·0001836	·0001391	·0001043	·0000774	·0000568	·0000413	2·2
2·3	·0002838	·0002208	·0001701	·0001297	·0000979	·0000731	·0000541	·0000396	·0000287	2·3
2·4	·0002003	·0001553	·0001193	·0000906	·0000682	·0000508	·0000374	·0000273	·0000197	2·4
2·5	·0001399	·0001081	·0000828	·0000627	·0000470	·0000349	·0000257	·0000186	·0000134	2·5
2·6	·0000967	·0000745^{+}	·0000568	·0000429	·0000321	·0000237	·0000174	·0000126	·0000090	2·6

k	d/N for $r = -\cdot35$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0074704	·0057037	·0043116	·0032267	·0023905^{+}	·0017532	·0012727	·0009144	·0006503	0·0
0·1	·0064276	·0048938	·0036888	·0027527	·0020334	·0014870	·0010762	·0007710	·0005467	0·1
0·2	·0054813	·0041614	·0031278	·0023272	·0017141	·0012497	·0009018	·0006441	·0004553	0·2
0·3	·0046323	·0035067	·0026279	·0019495^{+}	·0014316	·0010406	·0007486	·0005331	·0003756	0·3
0·4	·0038791	·0029278	·0021876	·0016180	·0011846	·0008584	·0006156	·0004370	·0003070	0·4
0·5	·0032183	·0024218	·0018040	·0013303	·0009709	·0007014	·0005015^{-}	·0003548	·0002485^{-}	0·5
0·6	·0026450^{+}	·0019844	·0014737	·0010833	·0007882	·0005676	·0004046	·0002854	·0001992	0·6
0·7	·0021533	·0016105^{-}	·0011923	·0008735^{-}	·0006337	·0004549	·0003232	·0002272	·0001581	0·7
0·8	·0017364	·0012945^{-}	·0009554	·0006979	·0005046	·0003611	·0002557	·0001792	·0001243	0·8
0·9	·0013863	·0010304	·0007581	·0005520	·0003978	·0002838	·0002003	·0001399	·0000967	0·9
1·0	·0010961	·0008121	·0005956	·0004323	·0003105^{+}	·0002208	·0001553	·0001081	·0000745^{+}	1·0
1·1	·0008582	·0006338	·0004633	·0003352	·0002400	·0001701	·0001193	·0000828	·0000568	1·1
1·2	·0006652	·0004897	·0003568	·0002573	·0001836	·0001297	·0000906	·0000627	·0000429	1·2
1·3	·0005105^{+}	·0003746	·0002720	·0001955^{+}	·0001391	·0000979	·0000682	·0000470	·0000321	1·3
1·4	·0003878	·0002836	·0002053	·0001471	·0001043	·0000731	·0000508	·0000349	·0000237	1·4
1·5	·0002916	·0002126	·0001534	·0001095^{+}	·0000774	·0000541	·0000374	·0000257	·0000174	1·5
1·6	·0002171	·0001577	·0001134	·0000807	·0000568	·0000396	·0000273	·0000186	·0000126	1·6
1·7	·0001600	·0001158	·0000830	·0000589	·0000413	·0000287	·0000197	·0000134	·0000090	1·7
1·8	·0001166	·0000842	·0000601	·0000425^{-}	·0000297	·0000206	·0000141	·0000095^{+}	·0000064	1·8
1·9	·0000842	·0000605^{+}	·0000431	·0000303	·0000212	·0000146	·0000100	·0000067	·0000045^{-}	1·9
2·0	·0000601	·0000431	·0000306	·0000215^{-}	·0000149	·0000102	·0000070	·0000047	·0000031	2·0
2·1	·0000425^{-}	·0000303	·0000215^{-}	·0000150^{-}	·0000104	·0000071	·0000048	·0000032	·0000021	2·1
2·2	·0000297	·0000212	·0000149	·0000104	·0000072	·0000049	·0000033	·0000022	·0000015^{-}	2·2
2·3	·0000206	·0000146	·0000102	·0000071	·0000049	·0000033	·0000022	·0000015^{-}	·0000010	2·3
2·4	·0000141	·0000100	·0000070	·0000048	·0000033	·0000022	·0000015^{+}	·0000010	·00000065^{+}	2·4
2·5	·00000095^{+}	·0000067	·0000047	·0000032	·0000022	·0000015^{-}	·0000010	·00000066	·00000043	2·5
2·6	·0000064	·0000045^{-}	·0000031	·0000021	·0000015^{-}	·0000010	·00000065^{+}	·00000043	·00000028	2·6

k	d/N for $r = -\cdot40$									k
	$h = 0\cdot0$	$h = 0\cdot1$	$h = 0\cdot2$	$h = 0\cdot3$	$h = 0\cdot4$	$h = 0\cdot5$	$h = 0\cdot6$	$h = 0\cdot7$	$h = 0\cdot8$	
0·0	·1845051	·1649375$^+$	·1462498	·1286015$^-$	·1121213	·0969041	·0830109	·0704688	·0592739	0·0
0·1	·1649375$^+$	·1470976	·1301130	·1141238	·0992408	·0855431	·0730781	·0618629	·0518859	0·1
0·2	·1462498	·1301130	·1147993	·1004304	·0870997	·0748716	·0637817	·0538376	·0450220	0·2
0·3	·1286015$^-$	·1141238	·1004304	·0876248	·0757849	·0649617	·0551797	·0464393	·0387180	0·3
0·4	·1121213	·0992408	·0870997	·0757849	·0653598	·0558634	·0473112	·0396968	·0329946	0·4
0·5	·0969041	·0855431	·0748716	·0649617	·0558634	·0476054	·0401955$^+$	·0336222	·0278577	0·5
0·6	·0830109	·0730781	·0637817	·0551797	·0473112	·0401955$^+$	·0338341	·0282120	·0233001	0·6
0·7	·0704688	·0618629	·0538376	·0464393	·0396968	·0336222	·0282120	·0234486	·0193028	0·7
0·8	·0592739	·0518859	·0450220	·0387180	·0329946	·0278577	·0233001	·0193028	·0158373	0·8
0·9	·0493939	·0431106	·0372953	·0319746	·0271625$^-$	·0228601	·0190578	·0157359	·0128673	0·9
1·0	·0407727	·0354798	·0305998	·0261523	·0221454	·0185770	·0154358	·0127023	·0103511	1·0
1·1	·0333350$^-$	·0289192	·0248638	·0211823	·0178787	·0149482	·0123788	·0101519	·0082441	1·1
1·2	·0269907	·0233427	·0200057	·0169884	·0142916	·0119090	·0098283	·0080323	·0065000$^-$	1·2
1·3	·0216403	·0186565$^-$	·0159379	·0134897	·0113103	·0093927	·0077249	·0062911	·0050728	1·3
1·4	·0171794	·0147632	·0125707	·0106043	·0088609	·0073332	·0060100	·0048771	·0039185$^+$	1·4
1·5	·0135022	·0115654	·0098152	·0082518	·0068715$^+$	·0056670	·0046280	·0037422	·0029957	1·5
1·6	·0105054	·0089689	·0075860	·0063559	·0052743	·0043344	·0035270	·0028416	·0022664	1·6
1·7	·0080910	·0068845$^-$	·0058032	·0048453	·0040067	·0032809	·0026601	·0021353	·0016967	1·7
1·8	·0061678	·0052303	·0043936	·0036556	·0030121	·0024576	·0019853	·0015878	·0012570	1·8
1·9	·0046534	·0039326	·0032920	·0027293	·0022408	·0018217	·0014662	·0011682	·0009213	1·9
2·0	·0034746	·0029262	·0024409	·0020164	·0016495$^+$	·0013360	·0010713	·0008504	·0006681	2·0
2·1	·0025673	·0021545$^+$	·0017908	·0014741	·0012014	·0009695$^+$	·0007745$^-$	·0006124	·0004793	2·1
2·2	·0018771	·0015697	·0013000	·0010662	·0008658	·0006960	·0005539	·0004364	·0003402	2·2
2·3	·0013580	·0011316	·0009338	·0007630	·0006173	·0004944	·0003919	·0003076	·0002389	2·3
2·4	·0009720	·0008070	·0006635$^+$	·0005401	·0004354	·0003474	·0002743	·0002144	·0001659	2·4
2·5	·0006884	·0005695$^-$	·0004665$^-$	·0003783	·0003038	·0002414	·0001899	·0001479	·0001140	2·5
2·6	·0004823	·0003975$^+$	·0003244	·0002621	·0002097	·0001660	·0001301	·0001009	·0000774	2·6

k	d/N for $r = -\cdot40$									k
	$h = 0\cdot9$	$h = 1\cdot0$	$h = 1\cdot1$	$h = 1\cdot2$	$h = 1\cdot3$	$h = 1\cdot4$	$h = 1\cdot5$	$h = 1\cdot6$	$h = 1\cdot7$	
0·0	·0493939	·0407727	·0333350$^-$	·0269907	·0216403	·0171794	·0135022	·0105054	·0080910	0·0
0·1	·0431106	·0354798	·0289192	·0233427	·0186565$^-$	·0147632	·0115654	·0089689	·0068845$^-$	0·1
0·2	·0372953	·0305998	·0248638	·0200057	·0159379	·0125707	·0098152	·0075860	·0058032	0·2
0·3	·0319746	·0261523	·0211823	·0169884	·0134897	·0106043	·0082518	·0063559	·0048453	0·3
0·4	·0271625$^-$	·0221454	·0178787	·0142916	·0113103	·0088609	·0068715$^+$	·0052743	·0040067	0·4
0·5	·0228601	·0185770	·0149482	·0119090	·0093927	·0073332	·0056670	·0043344	·0032809	0·5
0·6	·0190578	·0154358	·0123788	·0098283	·0077249	·0060100	·0046280	·0035270	·0026601	0·6
0·7	·0157359	·0127023	·0101519	·0080323	·0062911	·0048771	·0037422	·0028416	·0021353	0·7
0·8	·0128673	·0103511	·0082441	·0065000$^-$	·0050728	·0039185$^+$	·0029957	·0022664	·0016967	0·8
0·9	·0104184	·0083521	·0062286	·0052076	·0040496	·0031168	·0023740	·0017894	·0013347	0·9
1·0	·0083521	·0066720	·0052764	·0041303	·0032002	·0024539	·0018622	·0013984	·0010390	1·0
1·1	·0062286	·0052764	·0041576	·0032427	·0025032	·0019123	·0014457	·0010815$^+$	·0008005$^+$	1·1
1·2	·0052076	·0041303	·0032427	·0025198	·0019379	·0014749	·0011108	·0008278	·0006104	1·2
1·3	·0040496	·0032002	·0025032	·0019379	·0014848	·0011258	·0008446	·0006270	·0004605$^+$	1·3
1·4	·0031168	·0024539	·0019123	·0014749	·0011258	·0008503	·0006355$^-$	·0004699	·0003438	1·4
1·5	·0023740	·0018622	·0014457	·0011108	·0008446	·0006355$^-$	·0004731	·0003485$^-$	·0002539	1·5
1·6	·0017894	·0013984	·0010815$^+$	·0008278	·0006270	·0004699	·0003485$^-$	·0002557	·0001856	1·6
1·7	·0013347	·0010390	·0008005$^+$	·0006104	·0004605$^+$	·0003438	·0002539	·0001856	·0001342	1·7
1·8	·0009850$^-$	·0007639	·0005863	·0004453	·0003347	·0002488	·0001831	·0001332	·0000959	1·8
1·9	·0007192	·0005556	·0004248	·0003214	·0002406	·0001782	·0001305$^+$	·0000946	·0000679	1·9
2·0	·0005196	·0003998	·0003045$^-$	·0002295$^-$	·0001711	·0001262	·0000921	·0000665$^-$	·0000475$^-$	2·0
2·1	·0003713	·0002846	·0002159	·0001621	·0001203	·0000884	·0000643	·0000462	·0000329	2·1
2·2	·0002625$^+$	·0002004	·0001514	·0001132	·0000837	·0000613	·0000443	·0000318	·0000225$^+$	2·2
2·3	·0001836	·0001396	·0001051	·0000782	·0000576	·0000420	·0000303	·0000216	·0000152	2·3
2·4	·0001270	·0000962	·0000721	·0000535$^-$	·0000392	·0000280	·0000204	·0000145$^+$	·0000102	2·4
2·5	·0000869	·0000656	·0000489	·0000361	·0000264	·0000191	·0000137	·0000097	·0000068	2·5
2·6	·0000588	·0000442	·0000329	·0000242	·0000176	·0000127	·0000090	·0000063	·0000044	2·6

k	d/N for $r = -\cdot40$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0061678	·0046534	·0034746	·0025673	·0018771	·0013580	·0009720	·0006884	·0004823	0·0
0·1	·0052303	·0039326	·0029262	·0021545$^+$	·0015697	·0011316	·0008070	·0005695$^-$	·0003975$^+$	0·1
0·2	·0043936	·0032920	·0024409	·0017908	·0013000	·0009338	·0006635$^+$	·0004665$^-$	·0003244	0·2
0·3	·0036556	·0027293	·0020164	·0014741	·0010662	·0007630	·0005401	·0003783	·0002621	0·3
0·4	·0030121	·0022408	·0016495$^+$	·0012014	·0008658	·0006173	·0004354	·0003038	·0002097	0·4
0·5	·0024576	·0018217	·0013360	·0009695$^+$	·0006960	·0004944	·0003474	·0002414	·0001660	0·5
0·6	·0019853	·0014662	·0010713	·0007745$^-$	·0005539	·0003919	·0002743	·0001899	·0001301	0·6
0·7	·0015878	·0011682	·0008504	·0006124	·0004364	·0003076	·0002144	·0001479	·0001009	0·7
0·8	·0012570	·0009213	·0006681	·0004793	·0003402	·0002389	·0001659	·0001140	·0000774	0·8
0·9	·0009850$^-$	·0007192	·0005196	·0003713	·0002625$^+$	·0001836	·0001270	·0000869	·0000588	0·9
1·0	·0007639	·0005556	·0003998	·0002846	·0002004	·0001396	·0000962	·0000656	·0000442	1·0
1·1	·0005863	·0004248	·0003045$^-$	·0002159	·0001514	·0001051	·0000721	·0000489	·0000329	1·1
1·2	·0004453	·0003214	·0002295$^-$	·0001621	·0001132	·0000782	·0000535$^-$	·0000361	·0000242	1·2
1·3	·0003347	·0002406	·0001711	·0001203	·0000837	·0000576	·0000392	·0000264	·0000176	1·3
1·4	·0002488	·0001782	·0001262	·0000884	·0000613	·0000420	·0000280	·0000191	·0000127	1·4
1·5	·0001831	·0001305$^+$	·0000921	·0000643	·0000443	·0000303	·0000204	·0000137	·0000090	1·5
1·6	·0001332	·0000946	·0000665$^-$	·0000462	·0000318	·0000216	·0000145$^+$	·0000097	·0000063	1·6
1·7	·0000959	·0000679	·0000475$^-$	·0000329	·0000225$^+$	·0000152	·0000102	·0000068	·0000044	1·7
1·8	·0000683	·0000481	·0000336	·0000231	·0000158	·0000106	·0000071	·0000047	·0000030	1·8
1·9	·0000481	·0000338	·0000234	·0000161	·0000109	·0000073	·0000049	·0000032	·0000021	1·9
2·0	·0000336	·0000234	·0000162	·0000111	·0000075$^-$	·0000050$^+$	·0000033	·0000022	·0000014	2·0
2·1	·0000231	·0000161	·0000111	·0000075$^+$	·0000051	·0000034	·0000022	·0000014	·00000093	2·1
2·2	·0000158	·0000109	·0000075$^-$	·0000051	·0000034	·0000023	·0000015$^-$	·00000096	·00000061	2·2
2·3	·0000106	·0000073	·0000050$^+$	·0000034	·0000023	·0000015$^-$	·00000097	·00000063	·00000040	2·3
2·4	·0000071	·0000049	·0000033	·0000022	·0000015$^-$	·00000097	·00000063	·00000041	·00000026	2·4
2·5	·0000047	·0000032	·0000022	·0000014	·00000096	·00000063	·00000041	·00000026	·00000016	2·5
2·6	·0000030	·0000021	·0000014	·00000093	·00000061	·00000040	·00000026	·00000016	·00000010	2·6

k	d/N for $r = -\cdot45$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·1757120	·1561980	·1376688	·1202786	·1041466	·0893558	·0759529	·0639494	·0533250$^-$	0·0
0·1	·1561980	·1384562	·1216724	·1059795$^-$	·0914776	·0782330	·0662783	·0556144	·0462137	0·1
0·2	·1376688	·1216724	·1065976	·0925575$^-$	·0796341	·0678782	·0573101	·0479215$^-$	·0396794	0·2
0·3	·1202786	·1059795$^-$	·0925575$^-$	·0801067	·0686927	·0583523	·0490952	·0409059	·0337468	0·3
0·4	·1041466	·0914776	·0796341	·0686927	·0587040	·0496929	·0416600	·0345840	·0284250$^+$	0·4
0·5	·0893558	·0782330	·0678782	·0583523	·0496929	·0419144	·0350104	·0289554	·0237083	0·5
0·6	·0759529	·0662783	·0573101	·0490952	·0416600	·0350104	·0291344	·0240040	·0195779	0·6
0·7	·0639494	·0556144	·0479215$^-$	·0409059	·0345840	·0289554	·0240040	·0197002	·0160044	0·7
0·8	·0533250$^-$	·0462137	·0396794	·0337468	·0284250$^+$	·0237083	·0195779	·0160044	·0129498	0·8
0·9	·0440315$^-$	·0380241	·0325291	·0275627	·0231280	·0192155$^+$	·0158053	·0128686	·0103701	0·9
1·0	·0359976	·0309738	·0263994	·0222842	·0186265$^-$	·0154146	·0126782	·0102400	·0082177	1·0
1·1	·0291344	·0249759	·0212069	·0178322	·0148467	·0122375$^-$	·0099846	·0080629	·0064436	1·1
1·2	·0233403	·0199337	·0168606	·0141220	·0117108	·0096136	·0078114	·0062816	·0049987	1·2
1·3	·0185067	·0157451	·0132658	·0110669	·0091402	·0074725$^-$	·0060464	·0048417	·0038363	1·3
1·4	·0145219	·0123069	·0103280	·0085813	·0070583	·0057464	·0046300	·0036916	·0029124	1·4
1·5	·0112758	·0095183	·0079557	·0065832	·0053923	·0043715$^+$	·0035072	·0027843	·0021870	1·5
1·6	·0086629	·0072834	·0060629	·0049962	·0040752	·0032896	·0026278	·0020770	·0016242	1·6
1·7	·0065845$^+$	·0055136	·0045707	·0037508	·0030463	·0024485$^+$	·0019474	·0015324	·0011929	1·7
1·8	·0049511	·0041288	·0034084	·0027852	·0022524	·0018025$^-$	·0014273	·0011181	·0008665$^+$	1·8
1·9	·0036827	·0030583	·0025139	·0020455$^+$	·0016470	·0013123	·0010345$^-$	·0008068	·0006224	1·9
2·0	·0027094	·0022406	·0018340	·0014857	·0011911	·0009448	·0007414	·0005756	·0004420	2·0
2·1	·0019715$^+$	·0016235$^-$	·0013231	·0010672	·0008518	·0006726	·0005254	·0004061	·0003104	2·1
2·2	·0014188	·0011633	·0009440	·0007580	·0006023	·0004735$^-$	·0003682	·0002832	·0002155$^-$	2·2
2·3	·0010097	·0008243	·0006660	·0005324	·0004211	·0003295$^+$	·0002551	·0001953	·0001479	2·3
2·4	·0007106	·0005776	·0004646	·0003697	·0002911	·0002268	·0001747	·0001332	·0001004	2·4
2·5	·0004945$^+$	·0004002	·0003204	·0002539	·0001990	·0001543	·0001183	·0000897	·0000674	2·5
2·6	·0003403	·0002741	·0002185$^+$	·0001723	·0001344	·0001038	·0000792	·0000598	·0000446	2·6

k	d/N for $r = -\cdot45$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0440315⁻	·0359976	·0291344	·0233403	·0185067	·0145219	·0112758	·0086629	·0065845⁺	0·0
0·1	·0380241	·0309738	·0249759	·0199337	·0157451	·0123069	·0095183	·0072834	·0055136	0·1
0·2	·0325291	·0263994	·0212069	·0168606	·0132658	·0103280	·0079557	·0060629	·0045707	0·2
0·3	·0275627	·0222842	·0178322	·0141220	·0110669	·0085813	·0065832	·0049962	·0037508	0·3
0·4	·0231280	·0186265⁻	·0148467	·0117108	·0091402	·0070583	·0053923	·0040752	·0030463	0·4
0·5	·0192155⁺	·0154146	·0122375⁻	·0096136	·0074725⁻	·0057464	·0043715⁺	·0032896	·0024485⁺	0·5
0·6	·0158053	·0126282	·0099846	·0078114	·0060464	·0046300	·0035072	·0026278	·0019474	0·6
0·7	·0128686	·0102400	·0080629	·0062816	·0048417	·0036916	·0027843	·0020770	·0015324	0·7
0·8	·0103701	·0082177	·0064436	·0049987	·0038363	·0029124	·0021870	·0016242	·0011929	0·8
0·9	·0082700	·0065261	·0050954	·0039359	·0030075⁺	·0022732	·0016994	·0012565⁻	·0009187	0·9
1·0	·0065261	·0051281	·0039867	·0030661	·0023326	·0017552	·0013063	·0009615⁻	·0006998	1·0
1·1	·0050954	·0039867	·0030859	·0023629	·0017896	·0013406	·0009932	·0007277	·0005272	1·1
1·2	·0039359	·0030661	·0023629	·0018012	·0013581	·0010128	·0007469	·0005447	·0003928	1·2
1·3	·0030075⁺	·0023326	·0017896	·0013581	·0010194	·0007567	·0005555⁺	·0004032	·0002894	1·3
1·4	·0022732	·0017552	·0013406	·0010128	·0007567	·0005591	·0004086	·0002952	·0002109	1·4
1·5	·0016994	·0013063	·0009932	·0007469	·0005555⁺	·0004086	·0002971	·0002137	·0001519	1·5
1·6	·0012565⁻	·0009615⁻	·0007277	·0005447	·0004032	·0002952	·0002137	·0001529	·0001082	1·6
1·7	·0009187	·0006998	·0005272	·0003928	·0002894	·0002109	·0001519	·0001082	·0000761	1·7
1·8	·0006643	·0005037	·0003777	·0002801	·0002054	·0001490	·0001068	·0000757	·0000531	1·8
1·9	·0004749	·0003584	·0002675⁺	·0001975⁻	·0001441	·0001040	·0000742	·0000524	·0000365⁺	1·9
2·0	·0003357	·0002522	·0001874	·0001376	·0001000	·0000718	·0000510	·0000358	·0000248	2·0
2·1	·0002347	·0001754	·0001297	·0000948	·0000686	·0000490	·0000346	·0000242	·0000167	2·1
2·2	·0001622	·0001207	·0000888	·0000646	·0000465⁻	·0000331	·0000232	·0000162	·0000111	2·2
2·3	·0001108	·0000820	·0000601	·0000435⁺	·0000311	·0000220	·0000154	·0000107	·0000073	2·3
2·4	·0000748	·0000551	·0000402	·0000290	·0000206	·0000145⁺	·0000101	·0000070	·0000047	2·4
2·5	·0000499	·0000366	·0000266	·0000191	·0000135⁺	·0000095⁻	·0000066	·0000045⁻	·0000030	2·5
2·6	·0000330	·0000234	·0000174	·0000124	·0000087	·0000061	·0000042	·0000029	·0000019	2·6

k	d/N for $r = -\cdot45$								k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0049511	·0036827	·0027094	·0019715⁺	·0014188	·0010097	·0007106	·0004945⁺	·0003403	0·0
0·1	·0041288	·0030583	·0022406	·0016235⁻	·0011633	·0008243	·0005776	·0004002	·0002741	0·1
0·2	·0034084	·0025139	·0018340	·0013231	·0009440	·0006660	·0004646	·0003204	·0002185⁺	0·2
0·3	·0027852	·0020455⁺	·0014857	·0010672	·0007580	·0005324	·0003697	·0002539	·0001723	0·3
0·4	·0022524	·0016470	·0011911	·0008518	·0006023	·0004211	·0002911	·0001990	·0001344	0·4
0·5	·0018025⁻	·0013123	·0009448	·0006726	·0004735⁻	·0003295⁺	·0002268	·0001543	·0001038	0·5
0·6	·0014273	·0010345⁻	·0007414	·0005254	·0003682	·0002551	·0001747	·0001183	·0000792	0·6
0·7	·0011181	·0008068	·0005756	·0004061	·0002832	·0001953	·0001332	·0000897	·0000598	0·7
0·8	·0008665⁺	·0006224	·0004420	·0003104	·0002155⁻	·0001479	·0001004	·0000674	·0000446	0·8
0·9	·0006643	·0004749	·0003357	·0002347	·0001622	·0001108	·0000748	·0000499	·0000330	0·9
1·0	·0005037	·0003584	·0002522	·0001754	·0001207	·0000820	·0000551	·0000366	·0000234	1·0
1·1	·0003777	·0002675⁺	·0001874	·0001297	·0000888	·0000601	·0000402	·0000266	·0000174	1·1
1·2	·0002801	·0001975⁻	·0001376	·0000948	·0000646	·0000435⁺	·0000290	·0000191	·0000124	1·2
1·3	·0002054	·0001441	·0001000	·0000686	·0000465⁻	·0000311	·0000206	·0000135⁺	·0000087	1·3
1·4	·0001490	·0001040	·0000718	·0000490	·0000331	·0000220	·0000145⁺	·0000095⁻	·0000061	1·4
1·5	·0001068	·0000742	·0000510	·0000346	·0000232	·0000154	·0000101	·0000066	·0000042	1·5
1·6	·0000757	·0000524	·0000358	·0000242	·0000162	·0000107	·0000070	·0000045⁻	·0000029	1·6
1·7	·0000531	·0000365⁺	·0000248	·0000167	·0000111	·0000073	·0000047	·0000030	·0000019	1·7
1·8	·0000368	·0000252	·0000170	·0000114	·0000075⁺	·0000049	·0000032	·0000020	·0000013	1·8
1·9	·0000252	·0000172	·0000116	·0000077	·0000051	·0000033	·0000021	·0000013	·00000085⁻	1·9
2·0	·0000170	·0000116	·0000078	·0000051	·0000034	·0000022	·0000014	·00000088	·00000055⁺	2·0
2·1	·0000114	·0000077	·0000051	·0000034	·0000022	·0000014	·00000090	·00000057	·00000037	2·1
2·2	·0000075⁺	·0000051	·0000034	·0000022	·0000014	·00000092	·00000058	·00000036	·00000022	2·2
2·3	·0000049	·0000033	·0000022	·0000014	·00000092	·00000059	·00000037	·00000023	·00000014	2·3
2·4	·0000032	·0000021	·0000014	·00000090	·00000058	·00000037	·00000023	·00000014	·00000009	2·4
2·5	·0000020	·0000013	·00000088	·00000057	·00000036	·00000023	·00000014	·00000009	·00000005⁺	2·5
2·6	·0000013	·00000085⁻	·00000055⁺	·00000037	·00000022	·00000014	·00000009	·00000005⁺	·00000002	2·6

k	d/N for $r = -\cdot50$									k
	$h = 0\cdot0$	$h = 0\cdot1$	$h = 0\cdot2$	$h = 0\cdot3$	$h = 0\cdot4$	$h = 0\cdot5$	$h = 0\cdot6$	$h = 0\cdot7$	$h = 0\cdot8$	
0·0	·1666667	·1472109	·1288543	·1117443	·0959897	·0816598	·0687848	·0573588	·0473431	0·0
0·1	·1472109	·1295818	·1130216	·0976550+	·0835700	·0708178	·0594141	·0493418	·0405553	0·1
0·2	·1288543	·1130216	·0982164	·0845419	·0720665−	·0608253	·0508211	·0420282	·0343956	0·2
0·3	·1117443	·0976550+	·0845419	·0724876	·0615434	·0517301	·0430398	·0354399	·0288762	0·3
0·4	·0959897	·0835700	·0720665−	·0615434	·0520367	·0435548	·0360817	·0295794	·0239929	0·4
0·5	·0816598	·0708178	·0608253	·0517301	·0435548	·0362982	·0299375+	·0244321	·0197268	0·5
0·6	·0687848	·0594141	·0508211	·0430398	·0360817	·0299375+	·0245803	·0199680	·0160471	0·6
0·7	·0573588	·0493418	·0420282	·0354399	·0295794	·0244321	·0199680	·0161454	·0129134	0·7
0·8	·0473431	·0405553	·0343956	·0288762	·0239929	·0197268	·0160471	·0129134	·0102785+	0·8
0·9	·0386718	·0329853	·0278526	·0232782	·0192530	·0157559	·0127561	·0102154	·0080912	0·9
1·0	·0312570	·0265442	·0223135+	·0185637	·0152821	·0124469	·0100285−	·0079918	·0062984	1·0
1·1	·0249952	·0211319	·0176829	·0146428	·0119973	·0097244	·0077966	·0061823	·0048478	1·1
1·2	·0197727	·0166407	·0138601	·0114231	·0093142	·0075127	·0059935−	·0047286	·0036890	1·2
1·3	·0154710	·0129603	·0107439	·0088123	·0071504	·0057388	·0045553	·0035756	·0027751	1·3
1·4	·0119721	·0099821	·0082354	·0067219	·0054273	·0043340	·0034227	·0026728	·0020636	1·4
1·5	·0091615+	·0076023	·0062416	·0050694	·0040726	·0032357	·0025422	·0019748	·0015167	1·5
1·6	·0069322	·0057246	·0046769	·0037796	·0030210	·0023879	·0018663	·0014421	·0011017	1·6
1·7	·0051861	·0042617	·0034644	·0027856	·0022150−	·0017417	·0013541	·0010408	·0007909	1·7
1·8	·0038356	·0031363	·0025367	·0020292	·0016052	·0012556	·0009710	·0007424	·0005610	1·8
1·9	·0028042	·0022814	·0018359	·0014610	·0011497	·0008945+	·0006881	·0005232	·0003932	1·9
2·0	·0020265−	·0016403	·0013132	·0010396	·0008138	·0006298	·0004818	·0003644	·0002723	2·0
2·1	·0014474	·0011656	·0009283	·0007310	·0005692	·0004381	·0003334	·0002507	·0001862	2·1
2·2	·0010217	·0008185+	·0006484	·0005079	·0003934	·0003011	·0002279	·0001704	·0001260	2·2
2·3	·0007128	·0005680	·0004476	·0003487	·0002686	·0002045+	·0001539	·0001144	·0000841	2·3
2·4	·0004913	·0003895−	·0003053	·0002365+	·0001812	·0001372	·0001027	·0000759	·0000555+	2·4
2·5	·0003347	·0002639	·0002058	·0001586	·0001208	·0000909	·0000677	·0000498	·0000362	2·5
2·6	·0002253	·0001767	·0001370	·0001050−	·0000795−	·0000596	·0000441	·0000323	·0000233	2·6

k	d/N for $r = -\cdot50$									k
	$h = 0\cdot9$	$h = 1\cdot0$	$h = 1\cdot1$	$h = 1\cdot2$	$h = 1\cdot3$	$h = 1\cdot4$	$h = 1\cdot5$	$h = 1\cdot6$	$h = 1\cdot7$	
0·0	·0386718	·0312570	·0249952	·0197727	·0154710	·0119721	·0091615+	·0069322	·0051861	0·0
0·1	·0329853	·0265442	·0211319	·0166407	·0129603	·0099821	·0076023	·0057246	·0042617	0·1
0·2	·0278526	·0223135+	·0176829	·0138601	·0107439	·0082354	·0062416	·0046769	·0034644	0·2
0·3	·0232782	·0185637	·0146428	·0114231	·0088123	·0067219	·0050694	·0037796	·0027856	0·3
0·4	·0192530	·0152821	·0119973	·0093142	·0071504	·0054273	·0040726	·0030210	·0022150−	0·4
0·5	·0157559	·0124469	·0097244	·0075127	·0057388	·0043340	·0032357	·0023879	·0017417	0·5
0·6	·0127561	·0100285−	·0077966	·0059935−	·0045553	·0034227	·0025422	·0018663	·0013541	0·6
0·7	·0102154	·0079918	·0061823	·0047286	·0035756	·0026728	·0019748	·0014421	·0010408	0·7
0·8	·0080912	·0062984	·0048478	·0036890	·0027751	·0020636	·0015167	·0011017	·0007909	0·8
0·9	·0063376	·0049085+	·0037587	·0028455−	·0021294	·0015751	·0011515−	·0008319	·0005940	0·9
1·0	·0049085+	·0037823	·0028814	·0021699	·0016152	·0011884	·0008641	·0006209	·0004409	1·0
1·1	·0037587	·0028814	·0021836	·0016357	·0012111	·0008863	·0006409	·0004580	·0003234	1·1
1·2	·0028455−	·0021699	·0016357	·0012188	·0008975−	·0006532	·0004698	·0003339	·0002345−	1·2
1·3	·0021294	·0016152	·0012111	·0008975−	·0006574	·0004758	·0003403	·0002405+	·0001680	1·3
1·4	·0015751	·0011884	·0008863	·0006532	·0004758	·0003425+	·0002436	·0001712	·0001189	1·4
1·5	·0011515−	·0008641	·0006409	·0004698	·0003403	·0002436	·0001723	·0001204	·0000831	1·5
1·6	·0008319	·0006209	·0004580	·0003339	·0002405+	·0001712	·0001204	·0000837	·0000574	1·6
1·7	·0005940	·0004409	·0003234	·0002345−	·0001680	·0001189	·0000831	·0000574	·0000392	1·7
1·8	·0004190	·0003093	·0002257	·0001627	·0001159	·0000815+	·0000567	·0000389	·0000264	1·8
1·9	·0002921	·0002144	·0001556	·0001115+	·0000790	·0000553	·0000383	·0000261	·0000176	1·9
2·0	·0002012	·0001469	·0001059	·0000755−	·0000532	·0000370	·0000254	·0000173	·0000116	2·0
2·1	·0001369	·0000994	·0000713	·0000505+	·0000354	·0000245−	·0000167	·0000113	·0000075+	2·1
2·2	·0000920	·0000664	·0000474	·0000334	·0000232	·0000160	·0000108	·0000073	·0000048	2·2
2·3	·0000611	·0000439	·0000311	·0000218	·0000151	·0000103	·0000070	·0000046	·0000031	2·3
2·4	·0000401	·0000286	·0000201	·0000140	·0000097	·0000066	·0000044	·0000029	·0000019	2·4
2·5	·0000260	·0000184	·0000129	·0000089	·0000061	·0000042	·0000028	·0000018	·0000012	2·5
2·6	·0000166	·0000117	·0000082	·0000056	·0000039	·0000026	·0000017	·0000011	·00000072	2·6

k	d/N for $r = -\cdot50$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0038356	·0028042	·0020265$^-$	·0014474	·0010217	·0007128	·0004913	·0003347	·0002253	0·0
0·1	·0031363	·0022814	·0016403	·0011656	·0008185$^+$	·0005680	·0003895$^-$	·0002639	·0001767	0·1
0·2	·0025367	·0018359	·0013132	·0009283	·0006484	·0004476	·0003053	·0002058	·0001370	0·2
0·3	·0020292	·0014610	·0010396	·0007310	·0005079	·0003487	·0002365$^+$	·0001586	·0001050$^-$	0·3
0·4	·0016052	·0011497	·0008138	·0005692	·0003934	·0002686	·0001812	·0001208	·0000795$^-$	0·4
0·5	·0012556	·0008945$^+$	·0006298	·0004381	·0003011	·0002045$^+$	·0001372	·0000909	·0000596	0·5
0·6	·0009710	·0006881	·0004818	·0003334	·0002279	·0001539	·0001027	·0000677	·0000441	0·6
0·7	·0007424	·0005232	·0003644	·0002507	·0001704	·0001144	·0000759	·0000498	·0000323	0·7
0·8	·0005610	·0003932	·0002723	·0001864	·0001260	·0000841	·0000555$^+$	·0000362	·0000233	0·8
0·9	·0004190	·0002921	·0002012	·0001369	·0000920	·0000611	·0000401	·0000260	·0000166	0·9
1·0	·0003093	·0002144	·0001469	·0000994	·0000664	·0000439	·0000286	·0000184	·0000117	1·0
1·1	·0002257	·0001556	·0001059	·0000713	·0000474	·0000311	·0000201	·0000129	·0000082	1·1
1·2	·0001627	·0001115$^+$	·0000755$^-$	·0000505$^+$	·0000334	·0000218	·0000140	·0000089	·0000056	1·2
1·3	·0001159	·0000790	·0000532	·0000354	·0000232	·0000151	·0000097	·0000061	·0000039	1·3
1·4	·0000815$^+$	·0000553	·0000370	·0000245$^-$	·0000160	·0000103	·0000066	·0000042	·0000026	1·4
1·5	·0000567	·0000383	·0000254	·0000167	·0000108	·0000070	·0000044	·0000028	·0000017	1·5
1·6	·0000389	·0000261	·0000173	·0000113	·0000073	·0000046	·0000029	·0000018	·0000011	1·6
1·7	·0000264	·0000176	·0000116	·0000075$^+$	·0000048	·0000031	·0000019	·0000012	·00000072	1·7
1·8	·0000177	·0000117	·0000077	·0000050$^-$	·0000032	·0000020	·0000012	·00000076	·00000046	1·8
1·9	·0000117	·0000077	·0000050$^+$	·0000032	·0000020	·0000013	·00000079	·00000048	·00000025$^-$	1·9
2·0	·0000077	·0000050$^+$	·0000032	·0000021	·0000013	·00000081	·00000050$^+$	·00000029	·00000018	2·0
2·1	·0000050$^-$	·0000032	·0000021	·0000013	·00000082	·00000051	·00000031	·00000019	·00000011	2·1
2·2	·0000032	·0000020	·0000013	·00000082	·00000051	·00000031	·00000019	·00000011	·00000007	2·2
2·3	·0000020	·0000013	·00000081	·00000051	·00000031	·00000020	·00000012	·00000007	·00000004	2·3
2·4	·0000012	·00000079	·00000050$^+$	·00000031	·00000019	·00000012	·00000007	·00000004	·00000002	2·4
2·5	·00000076	·00000048	·00000029	·00000019	·00000011	·00000007	·00000004	·00000002	·00000001	2·5
2·6	·00000046	·00000025$^-$	·00000018	·00000011	·00000007	·00000004	·00000002	·00000001	·00000000	2·6

k	d/N for $r = -\cdot55$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·1573139	·1379225$^+$	·1197565$^-$	·1029553	·0876156	·0737906	·0614915$^-$	·0506918	·0413324	0·0
0·1	·1379225$^+$	·1204240	·1041166	·0891146	·0754916	·0632813	·0524796	·0430487	·0349227	0·1
0·2	·1197565$^-$	·1041166	·0896199	·0763569	·0643803	·0537065$^-$	·0443183	·0361695$^+$	·0291896	0·2
0·3	·1029553	·0891146	·0763569	·0647508	·0543306	·0450980	·0370254	·0300603	·0241304	0·3
0·4	·0876156	·0754916	·0643803	·0543306	·0453610	·0374610	·0305953	·0247076	·0197260	0·4
0·5	·0737906	·0632813	·0537065$^-$	·0450980	·0374610	·0307756	·0250014	·0200805$^+$	·0159430	0·5
0·6	·0614915$^-$	·0524796	·0443183	·0370254	·0305953	·0250014	·0202064	·0161344	·0127378	0·6
0·7	·0506918	·0430487	·0361695$^+$	·0300603	·0247076	·0200805$^+$	·0161344	·0128143	·0100586	0·7
0·8	·0413324	·0349227	·0291896	·0241304	·0197260	·0159430	·0127378	·0100586	·0078495$^+$	0·8
0·9	·0333271	·0280128	·0232900	·0191491	·0155672	·0125110	·0099384	·0078024	·0060528	0·9
1·0	·0265696	·0222148	·0183697	·0150202	·0121420	·0097023	·0076624	·0059799	·0046112	1·0
1·1	·0209407	·0174141	·0143206	·0116436	·0093586	·0074347	·0058368	·0045280	·0034704	1·1
1·2	·0163136	·0134918	·0110328	·0089192	·0071273	·0056287	·0043925$^+$	·0033869	·0025798	1·2
1·3	·0125604	·0103298	·0083991	·0067507	·0053626	·0042098	·0032653	·0025022	·0018942	1·3
1·4	·0095566	·0078148	·0063174	·0050477	·0039859	·0031101	·0023975$^+$	·0018258	·0013734	1·4
1·5	·0071843	·0058410	·0046942	·0037284	·0029263	·0022693	·0017386	·0013157	·0009834	1·5
1·6	·0053359	·0043129	·0034455$^-$	·0027201	·0021218	·0016353	·0012450$^-$	·0009362	·0006953	1·6
1·7	·0039150$^-$	·0031456	·0024979	·0019600	·0015195$^-$	·0011637	·0008803	·0006577	·0004853	1·7
1·8	·0028373	·0022660	·0017884	·0013947	·0010744	·0008176	·0006146	·0004563	·0003345$^-$	1·8
1·9	·0020309	·0016122	·0012645$^+$	·0009800	·0007502	·0005673	·0004237	·0003125$^-$	·0002276	1·9
2·0	·0014357	·0011327	·0008829	·0006799	·0005172	·0003886	·0002883	·0002112	·0001528	2·0
2·1	·0010022	·0007858	·0006087	·0004657	·0003520	·0002627	·0001937	·0001409	·0001013	2·1
2·2	·0006908	·0005382	·0004142	·0003150$^-$	·0002365$^-$	·0001754	·0001284	·0000928	·0000663	2·2
2·3	·0004702	·0003639	·0002784	·0002103	·0001569	·0001156	·0000841	·0000603	·0000428	2·3
2·4	·0003159	·0002429	·0001845$^+$	·0001386	·0001027	·0000752	·0000543	·0000387	·0000273	2·4
2·5	·0002092	·0001602	·0001209	·0000902	·0000663	·0000482	·0000346	·0000245$^+$	·0000172	2·5
2·6	·0001372	·0001042	·0000781	·0000579	·0000423	·0000306	·0000218	·0000149	·0000106	2·6

k	d/N for $r = -\cdot55$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0333271	·0265696	·0209407	·0163136	·0125604	·0095566	·0071843	·0053359	·0039150⁻	0·0
0·1	·0280128	·0222148	·0174141	·0134918	·0103298	·0078148	·0058410	·0043129	·0031456	0·1
0·2	·0232900	·0183697	·0143206	·0110328	·0083991	·0063174	·0046942	·0034455⁻	·0024979	0·2
0·3	·0191491	·0150202	·0116436	·0089192	·0067507	·0050477	·0037284	·0027201	·0019600	0·3
0·4	·0155672	·0121420	·0093586	·0071273	·0053626	·0039859	·0029263	·0021218	·0015195⁻	0·4
0·5	·0125110	·0097023	·0074347	·0056287	·0042098	·0031101	·0022693	·0016353	·0011637	0·5
0·6	·0099384	·0076624	·0058368	·0043925⁺	·0032653	·0023975⁺	·0017386	·0012450⁻	·0008803	0·6
0·7	·0078024	·0059799	·0045280	·0033869	·0025022	·0018258	·0013157	·0009362	·0006577	0·7
0·8	·0060528	·0046112	·0034704	·0025798	·0018942	·0013734	·0009834	·0006953	·0004853	0·8
0·9	·0046393	·0035129	·0026275⁺	·0019411	·0014163	·0010205⁻	·0007260	·0005099	·0003536	0·9
1·0	·0035129	·0026436	·0019651	·0014426	·0010458	·0007487	·0005292	·0003693	·0002544	1·0
1·1	·0026275⁺	·0019651	·0014515⁻	·0010588	·0007627	·0005424	·0003869	·0002641	·0001807	1·1
1·2	·0019411	·0014426	·0010588	·0007674	·0005492	·0003881	·0002707	·0001864	·0001267	1·2
1·3	·0014163	·0010458	·0007627	·0005492	·0003905⁻	·0002741	·0001899	·0001299	·0000877	1·3
1·4	·0010205⁻	·0007487	·0005424	·0003881	·0002741	·0001911	·0001316	·0000894	·0000600	1·4
1·5	·0007260	·0005292	·0003809	·0002707	·0001899	·0001316	·0000900	·0000607	·0000404	1·5
1·6	·0005099	·0003693	·0002641	·0001864	·0001299	·0000894	·0000607	·0000407	·0000269	1·6
1·7	·0003536	·0002544	·0001807	·0001267	·0000877	·0000600	·0000404	·0000269	·0000177	1·7
1·8	·0002421	·0001730	·0001221	·0000850⁺	·0000585⁻	·0000397	·0000266	·0000176	·0000115⁻	1·8
1·9	·0001636	·0001162	·0000814	·0000563	·0000385⁻	·0000259	·0000172	·0000113	·0000073	1·9
2·0	·0001091	·0000770	·0000536	·0000368	·0000250⁻	·0000167	·0000110	·0000072	·0000046	2·0
2·1	·0000719	·0000503	·0000348	·0000238	·0000160	·0000106	·0000070	·0000045⁺	·0000029	2·1
2·2	·0000467	·0000325⁻	·0000223	·0000151	·0000101	·0000066	·0000043	·0000028	·0000018	2·2
2·3	·0000300	·0000207	·0000141	·0000095⁻	·0000063	·0000041	·0000027	·0000017	·0000011	2·3
2·4	·0000191	·0000130	·0000088	·0000059	·0000039	·0000025⁺	·0000016	·0000010	·00000065⁻	2·4
2·5	·0000118	·0000081	·0000054	·0000036	·0000024	·0000015⁺	·00000097	·00000061	·00000038	2·5
2·6	·0000073	·0000049	·0000033	·0000022	·0000014	·00000091	·00000058	·00000036	·00000022	2·6

k	d/N for $r = -\cdot55$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0028373	·0020309	·0014357	·0010022	·0006908	·0004702	·0003159	·0002092	·0001372	0·0
0·1	·0022660	·0016122	·0011327	·0007858	·0005382	·0003639	·0002429	·0001602	·0001042	0·1
0·2	·0017884	·0012645⁺	·0008829	·0006087	·0004142	·0002784	·0001845⁺	·0001209	·0000781	0·2
0·3	·0013947	·0009800	·0006799	·0004657	·0003150⁻	·0002103	·0001386	·0000902	·0000579	0·3
0·4	·0010744	·0007502	·0005172	·0003520	·0002365⁻	·0001569	·0001027	·0000663	·0000423	0·4
0·5	·0008176	·0005673	·0003886	·0002627	·0001754	·0001156	·0000752	·0000482	·0000306	0·5
0·6	·0006146	·0004237	·0002883	·0001937	·0001284	·0000841	·0000543	·0000346	·0000218	0·6
0·7	·0004563	·0003125⁻	·0002112	·0001409	·0000928	·0000603	·0000387	·0000245⁺	·0000149	0·7
0·8	·0003345⁻	·0002276	·0001528	·0001013	·0000663	·0000428	·0000273	·0000172	·0000106	0·8
0·9	·0002421	·0001636	·0001091	·0000719	·0000467	·0000300	·0000191	·0000118	·0000073	0·9
1·0	·0001730	·0001162	·0000770	·0000503	·0000325⁻	·0000207	·0000130	·0000081	·0000049	1·0
1·1	·0001221	·0000814	·0000536	·0000348	·0000223	·0000141	·0000088	·0000054	·0000033	1·1
1·2	·0000850⁺	·0000563	·0000368	·0000238	·0000151	·0000095⁻	·0000059	·0000036	·0000022	1·2
1·3	·0000585⁻	·0000385⁻	·0000250⁻	·0000160	·0000101	·0000063	·0000039	·0000024	·0000014	1·3
1·4	·0000397	·0000259	·0000167	·0000106	·0000066	·0000041	·0000025⁺	·0000015⁺	·00000091	1·4
1·5	·0000266	·0000172	·0000110	·0000070	·0000043	·0000027	·0000016	·00000097	·00000058	1·5
1·6	·0000176	·0000113	·0000072	·0000045⁺	·0000028	·0000017	·0000010	·00000061	·00000036	1·6
1·7	·0000115⁻	·0000073	·0000046	·0000029	·0000018	·0000011	·00000065⁻	·00000038	·00000022	1·7
1·8	·0000074	·0000047	·0000029	·0000018	·0000011	·00000067	·00000040	·00000023	·00000013	1·8
1·9	·0000047	·0000030	·0000018	·0000011	·00000069	·00000041	·00000024	·00000015⁻	·00000008	1·9
2·0	·0000029	·0000018	·0000011	·00000070	·00000042	·00000025⁺	·00000015⁻	·00000008	·00000005⁻	2·0
2·1	·0000018	·0000011	·00000070	·00000042	·00000025⁺	·00000015⁻	·00000009	·00000005⁻	·00000003	2·1
2·2	·0000011	·00000069	·00000042	·00000025⁺	·00000015⁻	·00000009	·00000005⁻	·00000003	·00000002	2·2
2·3	·00000067	·00000041	·00000025⁺	·00000015⁻	·00000009	·00000005⁺	·00000003	·00000002	·00000001	2·3
2·4	·00000040	·00000024	·00000015⁻	·00000009	·00000005⁻	·00000003	·00000002	·00000001	·00000001	2·4
2·5	·00000023	·00000015⁻	·00000008	·00000005⁻	·00000003	·00000002	·00000001	·00000001	·00000000	2·5
2·6	·00000013	·00000008	·00000005⁻	·00000003	·00000002	·00000001	·00000001	·00000000		2·6

k	d/N for $r = -\cdot 60$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·1475836	·1282648	·1103130	·0938584	·0789827	·0657195^{+}	·0540578	·0439466	·0353029	0·0
0·1	·1282648	·1109204	·0949035^{-}	·0803157	·0672130	·0556079	·0454726	·0367452	·0293361	0·1
0·2	·1103130	·0949035^{-}	·0807649	·0679724	·0565592	·0465192	·0378115^{-}	·0303662	·0240902	0·2
0·3	·0938584	·0803157	·0679724	·0568800	·0470515^{+}	·0384660	·0310726	·0247964	·0195447	0·3
0·4	·0789827	·0672130	·0565592	·0470515^{+}	·0386867	·0314321	·0252301	·0200039	·0156633	0·4
0·5	·0657195^{+}	·0556079	·0465192	·0384660	·0314321	·0253763	·0202375^{+}	·0159396	·0123970	0·5
0·6	·0540578	·0454726	·0378115^{-}	·0310726	·0252301	·0202375^{+}	·0160328	·0125430	·0096885^{+}	0·6
0·7	·0439466	·0367452	·0303662	·0247964	·0200039	·0159396	·0125430	·0097456	·0074754	0·7
0·8	·0353029	·0293361	·0240902	·0195447	·0156633	·0123970	·0096885^{+}	·0074754	·0056935^{-}	0·8
0·9	·0280172	·0231352	·0188757	·0152133	·0121102	·0095193	·0073878	·0056599	·0042798	0·9
1·0	·0219629	·0180192	·0146050^{+}	·0116923	·0092438	·0072156	·0055604	·0042294	·0031749	1·0
1·1	·0170032	·0138586	·0111576	·0088714	·0069649	·0053984	·0041302	·0031188	·0023239	1·1
1·2	·0129980	·0105234	·0084147	·0066442	·0051795^{-}	·0039859	·0030274	·0022692	·0016783	1·2
1·3	·0098099	·0078883	·0062640	·0049112	·0038012	·0029040	·0021894	·0016288	·0011956	1·3
1·4	·0073086	·0058363	·0046020	·0035824	·0027526	·0020875^{-}	·0015621	·0011534	·0008402	1·4
1·5	·0053743	·0042616	·0033364	·0025784	·0019667	·0014803	·0010995^{-}	·0008057	·0005824	1·5
1·6	·0039001	·0030706	·0023866	·0018309	·0013862	·0010356	·0007633	·0005550^{+}	·0003981	1·6
1·7	·0027928	·0021830	·0016843	·0012826	·0009638	·0007145^{+}	·0005226	·0003771	·0002684	1·7
1·8	·0019732	·0015312	·0011726	·0008863	·0006609	·0004862	·0003529	·0002527	·0001784	1·8
1·9	·0013754	·0010594	·0008053	·0006040	·0004470	·0003263	·0002350^{-}	·0001669	·0001169	1·9
2·0	·0009458	·0007230	·0005455^{-}	·0004060	·0002981	·0002160	·0001543	·0001087	·0000755^{+}	2·0
2·1	·0006415^{-}	·0004867	·0003644	·0002691	·0001961	·0001410	·0000999	·0000698	·0000481	2·1
2·2	·0004291	·0003231	·0002400	·0001759	·0001272	·0000907	·0000638	·0000442	·0000302	2·2
2·3	·0002831	·0002115^{+}	·0001559	·0001134	·0000813	·0000575^{+}	·0000401	·0000276	·0000187	2·3
2·4	·0001842	·0001365^{+}	·0000999	·0000720	·0000513	·0000360	·0000249	·0000170	·0000114	2·4
2·5	·0001181	·0000869	·0000631	·0000451	·0000319	·0000222	·0000152	·0000103	·0000069	2·5
2·6	·0000747	·0000545^{+}	·0000393	·0000279	·0000195^{+}	·0000135^{-}	·0000092	·0000062	·0000041	2·6

k	d/N for $r = -\cdot 60$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·0280172	·0219629	·0170032	·0129980	·0098099	·0073086	·0053743	·0039001	·0027928	0·0
0·1	·0231352	·0180192	·0138586	·0105234	·0078883	·0058363	·0042616	·0030706	·0021830	0·1
0·2	·0188757	·0146050^{+}	·0111576	·0084147	·0062640	·0046020	·0033364	·0023866	·0016843	0·2
0·3	·0152133	·0116923	·0088714	·0066442	·0049112	·0035824	·0025784	·0018309	·0012826	0·3
0·4	·0121102	·0092438	·0069649	·0051795^{-}	·0038012	·0027526	·0019667	·0013862	·0009638	0·4
0·5	·0095193	·0072156	·0053984	·0039859	·0029040	·0020875^{-}	·0014803	·0010356	·0007145^{+}	0·5
0·6	·0073878	·0055604	·0041302	·0030274	·0021894	·0015621	·0010995^{-}	·0007633	·0005226	0·6
0·7	·0056599	·0042294	·0031188	·0022692	·0016288	·0011534	·0008057	·0005550^{+}	·0003771	0·7
0·8	·0042798	·0031749	·0023239	·0016783	·0011956	·0008402	·0005824	·0003981	·0002684	0·8
0·9	·0031938	·0023518	·0017086	·0012246	·0008658	·0006038	·0004153	·0002816	·0001884	0·9
1·0	·0023518	·0017189	·0012394	·0008816	·0006185^{-}	·0004280	·0002920	·0001965^{+}	·0001304	1·0
1·1	·0017086	·0012394	·0008869	·0006260	·0004358	·0002992	·0002025^{+}	·0001352	·0000890	1·1
1·2	·0012246	·0008816	·0006260	·0004384	·0003028	·0002062	·0001385^{+}	·0000917	·0000599	1·2
1·3	·0008658	·0006185^{-}	·0004358	·0003028	·0002075^{-}	·0001402	·0000934	·0000614	·0000397	1·3
1·4	·0006038	·0004280	·0002992	·0002062	·0001402	·0000940	·0000621	·0000405^{-}	·0000260	1·4
1·5	·0004153	·0002920	·0002025^{+}	·0001385^{+}	·0000934	·0000621	·0000407	·0000263	·0000168	1·5
1·6	·0002816	·0001965^{+}	·0001352	·0000917	·0000614	·0000405^{-}	·0000263	·0000169	·0000107	1·6
1·7	·0001884	·0001304	·0000890	·0000599	·0000397	·0000260	·0000168	·0000107	·0000067	1·7
1·8	·0001242	·0000853	·0000577	·0000385^{+}	·0000254	·0000164	·0000105^{+}	·0000066	·0000041	1·8
1·9	·0000807	·0000550^{+}	·0000369	·0000245^{+}	·0000160	·0000103	·0000065^{-}	·0000041	·0000025^{+}	1·9
2·0	·0000516	·0000350^{-}	·0000233	·0000153	·0000099	·0000063	·0000040	·0000025^{-}	·0000015^{+}	2·0
2·1	·0000327	·0000219	·0000145^{-}	·0000094	·0000060	·0000038	·0000024	·0000015^{-}	·00000089	2·1
2·2	·0000204	·0000135^{+}	·0000090	·0000057	·0000036	·0000023	·0000014	·00000086	·00000052	2·2
2·3	·0000125^{+}	·0000082	·0000054	·0000034	·0000022	·0000013	·00000082	·00000050^{+}	·00000030	2·3
2·4	·0000076	·0000050^{-}	·0000032	·0000020	·0000013	·00000078	·00000048	·00000029	·00000017	2·4
2·5	·0000045^{+}	·0000029	·0000019	·0000012	·00000073	·00000045^{-}	·00000027	·00000016	·00000010	2·5
2·6	·0000027	·0000017	·0000011	·00000067	·00000042	·00000025^{+}	·00000015^{+}	·00000009	·00000005^{+}	2·6

k	d/N for $r = -\cdot60$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·0019732	·0013754	·0009458	·0006415⁻	·0004291	·0002831	·0001842	·0001181	·0000747	0·0
0·1	·0015312	·0010594	·0007230	·0004867	·0003231	·0002115⁺	·0001365⁺	·0000869	·0000545⁺	0·1
0·2	·0011726	·0008053	·0005455⁻	·0003644	·0002400	·0001559	·0000999	·0000631	·0000393	0·2
0·3	·0008863	·0006040	·0004060	·0002691	·0001759	·0001134	·0000720	·0000451	·0000279	0·3
0·4	·0006609	·0004470	·0002981	·0001961	·0001272	·0000813	·0000513	·0000319	·0000195⁺	0·4
0·5	·0004862	·0003263	·0002160	·0001410	·0000907	·0000575⁺	·0000360	·0000222	·0000135⁻	0·5
0·6	·0003529	·0002350⁻	·0001543	·0000999	·0000638	·0000401	·0000249	·0000152	·0000092	0·6
0·7	·0002527	·0001669	·0001087	·0000698	·0000442	·0000276	·0000170	·0000103	·0000062	0·7
0·8	·0001784	·0001169	·0000755⁺	·0000481	·0000302	·0000187	·0000114	·0000069	·0000041	0·8
0·9	·0001242	·0000807	·0000516	·0000327	·0000204	·0000125⁺	·0000076	·0000045⁺	·0000027	0·9
1·0	·0000853	·0000550⁺	·0000350⁻	·0000219	·0000135⁺	·0000082	·0000049	·0000029	·0000017	1·0
1·1	·0000577	·0000369	·0000233	·0000145⁻	·0000090	·0000054	·0000032	·0000019	·0000011	1·1
1·2	·0000385⁺	·0000245⁺	·0000153	·0000094	·0000057	·0000034	·0000020	·0000012	·00000067	1·2
1·3	·0000254	·0000160	·0000099	·0000060	·0000036	·0000022	·0000013	·00000073	·00000042	1·3
1·4	·0000164	·0000103	·0000063	·0000038	·0000023	·0000013	·00000078	·00000045⁻	·00000025⁺	1·4
1·5	·0000105⁺	·0000065⁻	·0000040	·0000024	·0000014	·00000082	·00000048	·00000027	·00000015⁺	1·5
1·6	·0000066	·0000041	·0000025⁻	·0000015⁻	·00000086	·00000050⁺	·00000029	·00000016	·00000009	1·6
1·7	·0000041	·0000025⁺	·0000015⁺	·00000089	·00000052	·00000030	·00000017	·00000010	·00000005⁺	1·7
1·8	·0000025⁺	·0000015⁺	·00000091	·00000053	·00000031	·00000018	·00000010	·00000005⁺	·00000002	1·8
1·9	·0000015⁺	·00000091	·00000054	·00000031	·00000018	·00000010	·00000006	·00000003	·00000002	1·9
2·0	·00000091	·00000054	·00000032	·00000018	·00000010	·00000006	·00000003	·00000002	·00000001	2·0
2·1	·00000053	·00000031	·00000018	·00000011	·00000006	·00000003	·00000002	·00000001	·00000000	2·1
2·2	·00000031	·00000018	·00000010	·00000006	·00000003	·00000002	·00000001	·00000001		2·2
2·3	·00000018	·00000010	·00000006	·00000003	·00000002	·00000001	·00000001	·00000000		2·3
2·4	·00000010	·00000006	·00000003	·00000002	·00000001	·00000001	·00000000			2·4
2·5	·00000005⁺	·00000003	·00000002	·00000001	·00000000	·00000000				2·5
2·6	·00000002	·00000002	·00000001	·00000000						2·6

k	d/N for $r = -\cdot65$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·13738444	·11814907	·10044363	·08438644	·07004026	·05741457	·04647053	·03712815⁻	·02927486	0·0
0·1	·11814907	·10099015⁺	·08531418	·07120653	·05870105⁺	·04778352	·03839747	·03045178	·02382912	0·1
0·2	·10044363	·08531418	·07159960	·05935509	·04858909	·03926776	·03132135⁻	·02465184	·01914109	0·2
0·3	·08438644	·07120653	·05935509	·04886063	·03971028	·03185499	·02521607	·01969265⁺	·01516933	0·3
0·4	·07004026	·05870105⁺	·04858909	·03971028	·03203489	·02550301	·02003120	·01551943	·01185795⁺	0·4
0·5	·05741457	·04778352	·03926776	·03185499	·02550301	·02014534	·01569751	·01206339	·00914128	0·5
0·6	·04647053	·03839747	·03132135⁻	·02521607	·02003120	·01569751	·01213266	·00924691	·00694822	0·6
0·7	·03712815⁻	·03045178	·02465184	·01969265⁺	·01551943	·01206339	·00924691	·00698838	·00520632	0·7
0·8	·02927486	·02382912	·01914109	·01516933	·01185795⁺	·00914128	·00694822	·00520632	·00384510	0·8
0·9	·02277480	·01839480	·01465894	·01152332	·00893361	·00682912	·00514649	·00382288	·00279856	0·9
1·0	·01747795⁻	·01400509	·01107066	·00863094	·00663515⁺	·00502887	·00375699	·00276623	·00200701	1·0
1·1	·01322869	·01051473	·00824328	·00637283	·00485747	·00364969	·00270268	·00197225⁺	·00141806	1·1
1·2	·00987308	·00778314	·00605076	·00463800	·00350460	·00261009	·00191565⁺	·00138534	·00098699	1·2
1·3	·00726479	·00567916	·00437757	·00332650⁺	·00249156	·00183913	·00133767	·00095855⁻	·00067663	1·3
1·4	·00526934	·00408430	·00312109	·00235094	·00174523	·00127665⁺	·00092010	·00065326	·00045685⁻	1·4
1·5	·00376692	·00289463	·00219265⁻	·00163695⁺	·00120428	·00087293	·00062335⁻	·00043846	·00030375⁺	1·5
1·6	·00265370	·00202140	·00151762	·00112284	·00081855⁺	·00058788	·00041590	·00028980	·00019886	1·6
1·7	·00184202	·00139071	·00103476	·00075864	·00054797	·00038990	·00027325⁺	·00018860	·00012819	1·7
1·8	·00125967	·00094254	·00069494	·00050483	·00036126	·00025464	·00017677	·00012085⁻	·00008135⁻	1·8
1·9	·00084858	·00062919	·00045966	·00033082	·00023453	·00016375⁻	·00011259	·00007623	·00005082	1·9
2·0	·00056305⁺	·00041367	·00029941	·00021348	·00014991	·00010367	·00007060	·00004734	·00003125⁻	2·0
2·1	·00036794	·00026783	·00019205⁻	·00013563	·00009434	·00006462	·00004358	·00002894	·00001891	2·1
2·2	·00023678	·00017075⁻	·00012128	·00008484	·00005845⁻	·00003965⁻	·00002648	·00001741	·00001127	2·2
2·3	·00015004	·00010718	·00007541	·00005225⁻	·00003565⁻	·00002394	·00001583	·00001031	·00000661	2·3
2·4	·00009361	·00006624	·00004616	·00003167	·00002140	·00001423	·00000932	·00000601	·00000381	2·4
2·5	·00005750⁺	·00004030	·00002781	·00001890	·00001264	·00000833	·00000540	·00000345⁻	·00000216	2·5
2·6	·00003477	·00002413	·00001649	·00001110	·00000735⁺	·00000479	·00000308	·00000194	·00000121	2·6

k	d/N for $r = -\cdot65$									k
	$h = 0\cdot9$	$h = 1\cdot0$	$h = 1\cdot1$	$h = 1\cdot2$	$h = 1\cdot3$	$h = 1\cdot4$	$h = 1\cdot5$	$h = 1\cdot6$	$h = 1\cdot7$	
$0\cdot0$	·02277480	·01747795$^-$	·01322869	·00987308	·00726479	·00526934	·00376692	·00265370	·00184202	$0\cdot0$
$0\cdot1$	·01839480	·01400509	·01051473	·00778314	·00567916	·00408430	·00289463	·00202140	·00139071	$0\cdot1$
$0\cdot2$	·01465894	·01107066	·00824328	·00605076	·00437757	·00312109	·00219265$^-$	·00151762	·00103476	$0\cdot2$
$0\cdot3$	·01152332	·00863094	·00637283	·00463800	·00332650$^+$	·00235094	·00163695$^+$	·00112284	·00075864	$0\cdot3$
$0\cdot4$	·00893361	·00663515$^+$	·00485747	·00350460	·00249156	·00174523	·00120428	·00081855$^+$	·00054797	$0\cdot4$
$0\cdot5$	·00682912	·00502887	·00364969	·00261009	·00183913	·00127665$^+$	·00087293	·00058788	·00038990	$0\cdot5$
$0\cdot6$	·00514649	·00375699	·00270268	·00191565$^+$	·00133767	·00092010	·00062335$^-$	·00041590	·00027325$^+$	$0\cdot6$
$0\cdot7$	·00382288	·00276623	·00197225$^+$	·00138534	·00095855$^-$	·00065326	·00043846	·00028980	·00018860	$0\cdot7$
$0\cdot8$	·00279856	·00200701	·00141806	·00098699	·00067663	·00045685$^-$	·00030375$^+$	·00019886	·00012819	$0\cdot8$
$0\cdot9$	·00201873	·00143471	·00100446	·00069268	·00047045$^+$	·00031466	·00020723	·00013438	·00008579	$0\cdot9$
$1\cdot0$	·00143471	·00101035$^-$	·00070084	·00047881	·00032214	·00021342	·00013922	·00008941	·00005652	$1\cdot0$
$1\cdot1$	·00100446	·00070084	·00048163	·00032595$^+$	·00021722	·00014254	·00009208	·00005856	·00003666	$1\cdot1$
$1\cdot2$	·00069268	·00047881	·00032595$^+$	·00021851	·00014423	·00009373	·00005997	·00003777	·00002341	$1\cdot2$
$1\cdot3$	·00047045$^+$	·00032214	·00021722	·00014423	·00009428	·00006068	·00003844	·00002397	·00001471	$1\cdot3$
$1\cdot4$	·00031466	·00021342	·00014254	·00009373	·00006068	·00003867	·00002426	·00001498	·00000910	$1\cdot4$
$1\cdot5$	·000207230	·000139216	$\cdot0^492084$	$\cdot0^459965^+$	$\cdot0^438442$	$\cdot0^424258$	$\cdot0^415067$	$\cdot0^409211$	$\cdot0^4055414$	$1\cdot5$
$1\cdot6$	·000134377	$\cdot0^489405^+$	$\cdot0^458564$	$\cdot0^437665^+$	$\cdot0^423972$	$\cdot0^414978$	$\cdot0^409211$	$\cdot0^405574$	$\cdot0^4033201$	$1\cdot6$
$1\cdot7$	$\cdot0^485786$	$\cdot0^456523$	$\cdot0^436664$	$\cdot0^423411$	$\cdot0^414714$	$\cdot0^409102$	$\cdot0^4055414$	$\cdot0^4033201$	$\cdot0^419575^-$	$1\cdot7$
$1\cdot8$	$\cdot0^453913$	$\cdot0^435176$	$\cdot0^422593$	$\cdot0^414284$	$\cdot0^408888$	$\cdot0^405444$	$\cdot0^432808$	$\cdot0^419459$	$\cdot0^411357$	$1\cdot8$
$1\cdot9$	$\cdot0^433352$	$\cdot0^421547$	$\cdot0^413703$	$\cdot0^408577$	$\cdot0^405284$	$\cdot0^403204$	$\cdot0^419114$	$\cdot0^411222$	$\cdot0^406483$	$1\cdot9$
$2\cdot0$	$\cdot0^420308$	$\cdot0^412991$	$\cdot0^408180$	$\cdot0^405069$	$\cdot0^403091$	$\cdot0^418554$	$\cdot0^410958$	$\cdot0^406368$	$\cdot0^403641$	$2\cdot0$
$2\cdot1$	$\cdot0^412170$	$\cdot0^407708$	$\cdot0^404805^-$	$\cdot0^402948$	$\cdot0^401780$	$\cdot0^410573$	$\cdot0^406181$	$\cdot0^403555^+$	$\cdot0^402012$	$2\cdot1$
$2\cdot2$	$\cdot0^407178$	$\cdot0^404501$	$\cdot0^402778$	$\cdot0^401687$	$\cdot0^410081$	$\cdot0^405928$	$\cdot0^403430$	$\cdot0^401953$	$\cdot0^401094$	$2\cdot2$
$2\cdot3$	$\cdot0^404166$	$\cdot0^402586$	$\cdot0^415800$	$\cdot0^409498$	$\cdot0^405619$	$\cdot0^403271$	$\cdot0^401873$	$\cdot0^401055^+$	$\cdot0^400585^+$	$2\cdot3$
$2\cdot4$	$\cdot0^402380$	$\cdot0^401462$	$\cdot0^408843$	$\cdot0^405262$	$\cdot0^403081$	$\cdot0^401775^+$	$\cdot0^401006$	$\cdot0^400561$	$\cdot0^400308$	$2\cdot4$
$2\cdot5$	$\cdot0^401337$	$\cdot0^408136$	$\cdot0^404870$	$\cdot0^402868$	$\cdot0^401662$	$\cdot0^400948$	$\cdot0^400532$	$\cdot0^400293$	$\cdot0^400159$	$2\cdot5$
$2\cdot6$	$\cdot0^400740$	$\cdot0^404453$	$\cdot0^402639$	$\cdot0^401538$	$\cdot0^400882$	$\cdot0^400498$	$\cdot0^400276$	$\cdot0^400151$	$\cdot0^400811$	$2\cdot6$

k	d/N for $r = -\cdot65$									k
	$h = 1\cdot8$	$h = 1\cdot9$	$h = 2\cdot0$	$h = 2\cdot1$	$h = 2\cdot2$	$h = 2\cdot3$	$h = 2\cdot4$	$h = 2\cdot5$	$h = 2\cdot6$	
$0\cdot0$	·00125967	·00084858	·00056305$^+$	·00036794	·00023678	·00015004	·00009361	·00005750$^+$	·00003477	$0\cdot0$
$0\cdot1$	·00094254	·00062919	·00041367	·00026783	·00017075$^-$	·00010718	·00006624	·00004030	·00002413	$0\cdot1$
$0\cdot2$	·00069494	·00045966	·00029941	·00019205$^-$	·00012128	·00007541	·00004616	·00002781	·00001649	$0\cdot2$
$0\cdot3$	·00050483	·00033082	·00021348	·00013563	·00008484	·00005225$^-$	·00003167	·00001890	·00001110	$0\cdot3$
$0\cdot4$	·00036126	·00023453	·00014991	·00009434	·00005845$^-$	·00003565$^-$	·00002140	·00001264	·00000735$^+$	$0\cdot4$
$0\cdot5$	·000254641	·000163749	·000103673	$\cdot0^464618$	$\cdot0^439647$	$\cdot0^423944$	$\cdot0^414233$	$\cdot0^408327$	$\cdot0^404794$	$0\cdot5$
$0\cdot6$	·000176773	·000112592	$\cdot0^470600$	$\cdot0^443579$	$\cdot0^426478$	$\cdot0^415835^-$	$\cdot0^409329$	$\cdot0^405398$	$\cdot0^403077$	$0\cdot6$
$0\cdot7$	·000120847	$\cdot0^476232$	$\cdot0^447339$	$\cdot0^428936$	$\cdot0^417409$	$\cdot0^410308$	$\cdot0^406007$	$\cdot0^403445^+$	$\cdot0^401944$	$0\cdot7$
$0\cdot8$	$\cdot0^481347$	$\cdot0^450818$	$\cdot0^431250^-$	$\cdot0^418914$	$\cdot0^411267$	$\cdot0^406606$	$\cdot0^403811$	$\cdot0^402164$	$\cdot0^401209$	$0\cdot8$
$0\cdot9$	$\cdot0^453913$	$\cdot0^433352$	$\cdot0^420308$	$\cdot0^412170$	$\cdot0^407178$	$\cdot0^404166$	$\cdot0^402380$	$\cdot0^401337$	$\cdot0^400740$	$0\cdot9$
$1\cdot0$	$\cdot0^435176$	$\cdot0^421547$	$\cdot0^412991$	$\cdot0^407708$	$\cdot0^404501$	$\cdot0^402586$	$\cdot0^401462$	$\cdot0^408136$	$\cdot0^404453$	$1\cdot0$
$1\cdot1$	$\cdot0^422593$	$\cdot0^413703$	$\cdot0^408180$	$\cdot0^404805^-$	$\cdot0^402778$	$\cdot0^415800$	$\cdot0^408843$	$\cdot0^404870$	$\cdot0^402639$	$1\cdot1$
$1\cdot2$	$\cdot0^414284$	$\cdot0^408577$	$\cdot0^405069$	$\cdot0^402948$	$\cdot0^401687$	$\cdot0^409498$	$\cdot0^405262$	$\cdot0^402868$	$\cdot0^401538$	$1\cdot2$
$1\cdot3$	$\cdot0^408888$	$\cdot0^405284$	$\cdot0^403091$	$\cdot0^401780$	$\cdot0^410081$	$\cdot0^405619$	$\cdot0^403081$	$\cdot0^401662$	$\cdot0^400882$	$1\cdot3$
$1\cdot4$	$\cdot0^405444$	$\cdot0^403204$	$\cdot0^418554$	$\cdot0^410573$	$\cdot0^405928$	$\cdot0^403271$	$\cdot0^401775^+$	$\cdot0^400948$	$\cdot0^400498$	$1\cdot4$
$1\cdot5$	$\cdot0^432808$	$\cdot0^419114$	$\cdot0^410958$	$\cdot0^406181$	$\cdot0^403430$	$\cdot0^401873$	$\cdot0^401006$	$\cdot0^400532$	$\cdot0^400276$	$1\cdot5$
$1\cdot6$	$\cdot0^419459$	$\cdot0^411222$	$\cdot0^406368$	$\cdot0^403555^+$	$\cdot0^401953$	$\cdot0^401055^+$	$\cdot0^400561$	$\cdot0^400293$	$\cdot0^400151$	$1\cdot6$
$1\cdot7$	$\cdot0^411357$	$\cdot0^406483$	$\cdot0^403641$	$\cdot0^402012$	$\cdot0^401094$	$\cdot0^400585^+$	$\cdot0^400308$	$\cdot0^400159$	$\cdot0^400811$	$1\cdot7$
$1\cdot8$	$\cdot0^406522$	$\cdot0^403685^-$	$\cdot0^402048$	$\cdot0^401120$	$\cdot0^400603$	$\cdot0^400319$	$\cdot0^400166$	$\cdot0^400851$	$\cdot0^400429$	$1\cdot8$
$1\cdot9$	$\cdot0^403685^-$	$\cdot0^402061$	$\cdot0^401134$	$\cdot0^400614$	$\cdot0^400327$	$\cdot0^400172$	$\cdot0^400882$	$\cdot0^400447$	$\cdot0^400223$	$1\cdot9$
$2\cdot0$	$\cdot0^402048$	$\cdot0^401134$	$\cdot0^400617$	$\cdot0^400330$	$\cdot0^400174$	$\cdot0^400090$	$\cdot0^400046$	$\cdot0^400023$	$\cdot0^400011$	$2\cdot0$
$2\cdot1$	$\cdot0^401120$	$\cdot0^400614$	$\cdot0^400330$	$\cdot0^400175$	$\cdot0^400091$	$\cdot0^400046$	$\cdot0^400023$	$\cdot0^400011$	$\cdot0^400005$	$2\cdot1$
$2\cdot2$	$\cdot0^400603$	$\cdot0^400327$	$\cdot0^400174$	$\cdot0^400091$	$\cdot0^400047$	$\cdot0^400023$	$\cdot0^400011$	$\cdot0^400005$	$\cdot0^400002$	$2\cdot2$
$2\cdot3$	$\cdot0^400319$	$\cdot0^400172$	$\cdot0^400090$	$\cdot0^400046$	$\cdot0^400023$	$\cdot0^400011$	$\cdot0^400005$	$\cdot0^400002$	$\cdot0^400001$	$2\cdot3$
$2\cdot4$	$\cdot0^400166$	$\cdot0^400089$	$\cdot0^400046$	$\cdot0^400023$	$\cdot0^400011$	$\cdot0^400005$	$\cdot0^400002$	$\cdot0^400001$	$\cdot0^400000$	$2\cdot4$
$2\cdot5$	$\cdot0^400085$	$\cdot0^400045$	$\cdot0^400023$	$\cdot0^400011$	$\cdot0^400005$	$\cdot0^400002$	$\cdot0^400001$	$\cdot0^400000$	$\cdot0^400000$	$2\cdot5$
$2\cdot6$	$\cdot0^400042$	$\cdot0^400022$	$\cdot0^400011$	$\cdot0^400005$	$\cdot0^400002$	$\cdot0^400001$	$\cdot0^400000$	$\cdot0^400000$	$\cdot0^400000$	$2\cdot6$

0^4 indicates that four zeros must be placed before the figures that follow.

k	d/N for $r = -\cdot70$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·12659166	·10745519	·09004105^{+}	·07445180	·06072555^{+}	·04884035^{+}	·03872180	·03025303	·02328595^{-}	0·0
0·1	·10745519	·09052528	·07526022	·06172350^{+}	·04991962	·03979999	·03127159	·02420725^{-}	·01845654	0·1
0·2	·09004105^{+}	·07526022	·06205979	·05046818	·04046128	·03196965^{+}	·02488758	·01908328	·01440906	0·2
0·3	·07445180	·06172350^{+}	·05046818	·04068415^{+}	·03232451	·02530494	·01951290	·01481720	·01107722	0·3
0·4	·06072555^{+}	·04991962	·04046128	·03232451	·02544562	·01973133	·01506763	·01132840	·00838350^{-}	0·4
0·5	·04884035^{+}	·03979999	·03196965^{+}	·02530494	·01973133	·01515204	·01145612	·00852612	·00624477	0·5
0·6	·03872180	·03127159	·02488758	·01951290	·01506763	·01145612	·00857420	·00631560	·00457729	0·6
0·7	·03025303	·02420725^{-}	·01908328	·01481720	·01132840	·00852612	·00631560	·00460324	·00330076	0·7
0·8	·02328595^{-}	·01845654	·01440906	·01107722	·00838350^{-}	·00624477	·00457729	·00330076	·00234126	0·8
0·9	·01765282	·01385649	·01071087	·00815108	·00610548	·00450030	·00326355^{-}	·00232799	·00163320	0·9
1·0	·01317710	·01024119	·00783649	·00590236	·00437486	·00319039	·00228866	·00161471	·00112024	1·0
1·1	·00968298	·00744980	·00564202	·00420509	·00308372	·00222456	·00157836	·00110124	·00075544	1·1
1·2	·00700303	·00533269	·00399647	·00294702	·00213784	·00152536	·00107028	·00073838	·00050078	1·2
1·3	·00498381	·00375551	·00278464	·00203130	·00145745^{+}	·00102839	·00071349	·00048666	·00032628	1·3
1·4	·00348941	·00260157	·00190826	·00137681	·00097694	·00068161	·00046755^{-}	·00031525^{+}	·00020892	1·4
1·5	·00240314	·00177243	·00128591	·00091753	·00064376	·00044408	·00030113	·00020070	·00013146	1·5
1·6	·00162768	·00118742	·00085197	·00060111	·00041698	·00028435^{+}	·00019059	·00012555^{-}	·00008127	1·6
1·7	·00108407	·00078212	·00055490	·00038709	·00026545^{+}	·00017893	·00011854	·00007717	·00004936	1·7
1·8	·00070987	·00050643	·00035525^{-}	·00024499	·00016607	·00011064	·00007243	·00004660	·00002945^{-}	1·8
1·9	·00045695^{-}	·00032232	·00022352	·00015237	·00010208	·00006721	·00004348	·00002764	·00001726	1·9
2·0	·00028912	·00020161	·00013821	·00009312	·00006166	·00004011	·00002564	·0^{4}16104	·0^{4}09935^{-}	2·0
2·1	·00017978	·00012393	·00008397	·00005591	·00003658	·00002352	·0^{4}14854	·0^{4}09216	·0^{4}05616	2·1
2·2	·00010986	·00007485^{+}	·00005012	·00003298	·00002132	·0^{4}13544	·0^{4}08451	·0^{4}05180	·0^{4}03118	2·2
2·3	·00006596	·00004442	·00002939	·00001911	·0^{4}12207	·0^{4}07660	·0^{4}04722	·0^{4}02859	·0^{4}01700	2·3
2·4	·00003891	·00002589	·00001693	·0^{4}10876	·0^{4}06864	·0^{4}04255^{+}	·0^{4}02591	·0^{4}01549	·0^{4}00910	2·4
2·5	·00002255^{+}	·00001483	·0^{4}09579	·0^{4}06080	·0^{4}03791	·0^{4}02321	·0^{4}01396	·0^{4}00825^{-}	·0^{4}00478	2·5
2·6	·00001284	·0^{4}08340	·0^{4}05323	·0^{4}03338	·0^{4}02056	·0^{4}01243	·0^{4}00739	·0^{4}00431	·0^{4}00247	2·6

k	d/N for $r = -\cdot70$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·01765282	·01317710	·00968298	·00700303	·00498381	·00348941	·00240314	·00162768	·00108407	0·0
0·1	·01385649	·01024119	·00744980	·00533269	·00375551	·00260157	·00177243	·00118742	·00078212	0·1
0·2	·01071087	·00783649	·00564202	·00399647	·00278464	·00190826	·00128591	·00085197	·00055490	0·2
0·3	·00815108	·00590236	·00420509	·00294702	·00203130	·00137681	·00091753	·00060111	·00038709	0·3
0·4	·00610548	·00437486	·00308372	·00213784	·00145745^{+}	·00097694	·00064376	·00041698	·00026545^{+}	0·4
0·5	·00450030	·00319039	·00222456	·00152536	·00102839	·00068161	·00044408	·00028435^{+}	·00017893	0·5
0·6	·00326355^{-}	·00228866	·00157836	·00107028	·00071349	·00046755^{-}	·00030113	·00019059	·00011854	0·6
0·7	·00232799	·00161471	·00110124	·00073838	·00048666	·00031525^{+}	·00020070	·00012555^{-}	·00007717	0·7
0·8	·00163320	·00112024	·00075544	·00050078	·00032628	·00020892	·00013146	·00008127	·00004936	0·8
0·9	·00112665^{-}	·00076413	·00050946	·00033385^{+}	·00021501	·00013607	·00008461	·00005169	·00003102	0·9
1·0	·00076413	·00051238	·00033770	·00021874	·00013923	·00008708	·00005350^{+}	·00003230	·0^{4}19149	1·0
1·1	·00050946	·00033770	·00022000	·00014084	·00008859	·00005475^{+}	·00003324	·0^{4}19824	·0^{4}11612	1·1
1·2	·00033385^{+}	·00021874	·00014084	·00008910	·00005539	·00003382	·0^{4}20287	·0^{4}11952	·0^{4}06196	1·2
1·3	·00021501	·00013923	·00008859	·00005539	·00003402	·0^{4}20522	·0^{4}12161	·0^{4}07078	·0^{4}04046	1·3
1·4	·00013607	·00008708	·00005475^{+}	·00003382	·0^{4}20522	·0^{4}12232	·0^{4}07160	·0^{4}04116	·0^{4}02324	1·4
1·5	·00008461	·00005350^{+}	·00003324	·0^{4}20287	·0^{4}12161	·0^{4}07160	·0^{4}04141	·0^{4}02351	·0^{4}01311	1·5
1·6	·00005169	·00003230	·0^{4}19824	·0^{4}11952	·0^{4}07078	·0^{4}04116	·0^{4}02351	·0^{4}01319	·0^{4}00726	1·6
1·7	·00003102	·0^{4}19149	·0^{4}11612	·0^{4}06916	·0^{4}04046	·0^{4}02324	·0^{4}01311	·0^{4}00726	·0^{4}00395^{-}	1·7
1·8	·0^{4}18285^{-}	·0^{4}11152	·0^{4}06680	·0^{4}03930	·0^{4}02271	·0^{4}01288	·0^{4}00718	·0^{4}00393	·0^{4}00211	1·8
1·9	·0^{4}10587	·0^{4}06378	·0^{4}03774	·0^{4}02193	·0^{4}01252	·0^{4}00701	·0^{4}00386	·0^{4}00208	·0^{4}001105^{+}	1·9
2·0	·0^{4}06020	·0^{4}03583	·0^{4}02094	·0^{4}01202	·0^{4}00677	·0^{4}00375^{-}	·0^{4}00204	·0^{4}001086	·0^{4}000569	2·0
2·1	·0^{4}03362	·0^{4}01976	·0^{4}01141	·0^{4}00647	·0^{4}00360	·0^{4}00197	·0^{4}001055^{-}	·0^{4}000556	·0^{4}000287	2·1
2·2	·0^{4}01844	·0^{4}01070	·0^{4}00610	·0^{4}00342	·0^{4}00188	·0^{4}001013	·0^{4}000537	·0^{4}000279	·0^{4}000142	2·2
2·3	·0^{4}00993	·0^{4}00569	·0^{4}00320	·0^{4}00177	·0^{4}000961	·0^{4}000512	·0^{4}000268	·0^{4}000137	·0^{4}000069	2·3
2·4	·0^{4}00525^{-}	·0^{4}00297	·0^{4}00165^{+}	·0^{4}000902	·0^{4}000483	·0^{4}000254	·0^{4}000131	·0^{4}000066	·0^{4}000033	2·4
2·5	·0^{4}00272	·0^{4}00152	·0^{4}000836	·0^{4}000451	·0^{4}000239	·0^{4}000124	·0^{4}000063	·0^{4}000032	·0^{4}000015^{+}	2·5
2·6	·0^{4}00139	·0^{4}000766	·0^{4}000415^{+}	·0^{4}000221	·0^{4}000115^{+}	·0^{4}000059	·0^{4}000030	·0^{4}000015^{-}	·0^{4}000007	2·6

0^{4} indicates that four zeros must be placed before the figures that follow.

k	d/N for $r = -\cdot70$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·00070987	·00045695⁻	·00028912	·00017978	·00010986	·00006596	·00003891	·00002255⁺	·00001284	0·0
0·1	·00050643	·00032232	·00020161	·00012393	·00007485⁺	·00004442	·00002589	·00001483	·0⁴08340	0·1
0·2	·00035525⁻	·00022352	·00013821	·00008397	·00005012	·00002939	·00001693	·0⁴09579	·0⁴05323	0·2
0·3	·00024499	·00015237	·00009312	·00005591	·00003298	·00001911	·0⁴10876	·0⁴06080	·0⁴03338	0·3
0·4	·00016607	·00010208	·00006166	·00003658	·00002132	·0⁴12207	·0⁴06864	·0⁴03791	·0⁴02056	0·4
0·5	·00011064	·00006721	·00004011	·00002352	·0⁴13544	·0⁴07660	·0⁴04255⁺	·0⁴02321	·0⁴01243	0·5
0·6	·00007243	·00004348	·00002564	·0⁴14854	·0⁴08451	·0⁴04722	·0⁴02591	·0⁴01396	·0⁴00739	0·6
0·7	·00004660	·00002764	·000016104	·0⁴09216	·0⁴05180	·0⁴02859	·0⁴01549	·0⁴00825⁻	·0⁴00431	0·7
0·8	·00002945⁻	·0⁴17259	·0⁴09935⁻	·0⁴05616	·0⁴03118	·0⁴01700	·0⁴00910	·0⁴00478	·0⁴00247	0·8
0·9	·0⁴18285⁻	·0⁴10587	·0⁴06020	·0⁴03362	·0⁴01844	·0⁴00993	·0⁴00525⁻	·0⁴00272	·0⁴00139	0·9
1·0	·0⁴11152	·0⁴06378	·0⁴03583	·0⁴01976	·0⁴01070	·0⁴00569	·0⁴00297	·0⁴00152	·0⁴000766	1·0
1·1	·0⁴06680	·0⁴03774	·0⁴02094	·0⁴01141	·0⁴00610	·0⁴00320	·0⁴00165⁺	·0⁴000836	·0⁴000415⁺	1·1
1·2	·0⁴03930	·0⁴02193	·0⁴01202	·0⁴00647	·0⁴00342	·0⁴00177	·0⁴000902	·0⁴000451	·0⁴000221	1·2
1·3	·0⁴02271	·0⁴01252	·0⁴00677	·0⁴00360	·0⁴00188	·0⁴000961	·0⁴000483	·0⁴000239	·0⁴000115⁺	1·3
1·4	·0⁴01288	·0⁴00701	·0⁴00375⁻	·0⁴00197	·0⁴001013	·0⁴000512	·0⁴000254	·0⁴000124	·0⁴000059	1·4
1·5	·0⁴00718	·0⁴00386	·0⁴00204	·0⁴001055⁻	·0⁴000537	·0⁴000268	·0⁴000131	·0⁴000063	·0⁴000030	1·5
1·6	·0⁴00393	·0⁴00208	·0⁴001086	·0⁴000556	·0⁴000279	·0⁴000137	·0⁴000066	·0⁴000032	·0⁴000015⁻	1·6
1·7	·0⁴00211	·0⁴001105⁺	·0⁴000569	·0⁴000287	·0⁴000142	·0⁴000069	·0⁴000033	·0⁴000015⁺	·0⁴000007	1·7
1·8	·0⁴001111	·0⁴000575⁺	·0⁴000292	·0⁴000146	·0⁴000071	·0⁴000034	·0⁴000016	·0⁴000007	·0⁴000003	1·8
1·9	·0⁴000575⁺	·0⁴000294	·0⁴000147	·0⁴000072	·0⁴000035⁺	·0⁴000016	·0⁴000008	·0⁴000003	·0⁴000002	1·9
2·0	·0⁴000292	·0⁴000147	·0⁴000073	·0⁴000035⁺	·0⁴000017	·0⁴000008	·0⁴000004	·0⁴000002	·0⁴000001	2·0
2·1	·0⁴000146	·0⁴000072	·0⁴000035⁺	·0⁴000017	·0⁴000008	·0⁴000004	·0⁴000002	·0⁴000001		2·1
2·2	·0⁴000071	·0⁴000035⁺	·0⁴000017	·0⁴000008	·0⁴000004	·0⁴000002	·0⁴000001			2·2
2·3	·0⁴000034	·0⁴000016	·0⁴000008	·0⁴000004	·0⁴000002	·0⁴000001				2·3
2·4	·0⁴000016	·0⁴000008	·0⁴000004	·0⁴000002	·0⁴000001					2·4
2·5	·0⁴000007	·0⁴000003	·0⁴000002	·0⁴000001						2·5
2·6	·0⁴000003	·0⁴000002	·0⁴000001							2·6

k	d/N for $r = -\cdot75$									k
	$h = 0{\cdot}0$	$h = 0{\cdot}1$	$h = 0{\cdot}2$	$h = 0{\cdot}3$	$h = 0{\cdot}4$	$h = 0{\cdot}5$	$h = 0{\cdot}6$	$h = 0{\cdot}7$	$h = 0{\cdot}8$	
0·0	·11502673	·09601193	·07895514	·06393679	·05096068	·03996170	·03081764	·02336334	·01740584	0·0
0·1	·09601193	·07937492	·06462260	·05178732	·04083268	·03166336	·02413803	·01808359	·01330939	0·1
0·2	·07895514	·06462260	·05206585⁻	·04127527	·03218190	·02466872	·01858381	·01375392	·00999731	0·2
0·3	·06393679	·05178732	·04127527	·03235663	·02493843	·01889056	·01405849	·01027567	·00737442	0·3
0·4	·05096068	·04083268	·03218190	·02493843	·01899393	·01421330	·01044640	·00753871	·00534026	0·4
0·5	·03996170	·03166336	·02466872	·01889056	·01421330	·01050393	·00762221	·00542947	·00379547	0·5
0·6	·03081764	·02413803	·01858381	·01405849	·01044640	·00762221	·00545954	·00383772	·00264682	0·6
0·7	·02336334	·01808359	·01375392	·01027567	·00753871	·00542947	·00383772	·00266155⁻	·00181066	0·7
0·8	·01740584	·01330939	·00999731	·00737442	·00534026	·00379547	·00264682	·00181066	·00121481	0·8
0·9	·01273898	·00962023	·00713469	·00519483	·00371236	·00260314	·00179065⁻	·00120807	·00079919	0·9
1·0	·00915625⁺	·00682714	·00499785⁺	·00359109	·00253193	·00175128	·00118805⁺	·00079032	·00051544	1·0
1·1	·00646129	·00475554	·00343554	·00243550⁺	·00169384	·00115543	·00077289	·00050687	·00032585⁻	1·1
1·2	·00447528	·00325055⁺	·00231690	·00162018	·00111127	·00074745⁺	·00049291	·00031864	·00020188	1·2
1·3	·00304167	·00217976	·00153259	·00105695⁺	·00071483	·00047401	·00030812	·00019631	·00012256	1·3
1·4	·00202812	·00143371	·00099416	·00067606	·00045077	·00029464	·00018876	·00011851	·00007290	1·4
1·5	·00132639	·00092475⁻	·00063230	·00042391	·00027861	·00017948	·00011331	·00007009	·00004248	1·5
1·6	·00085066	·00058481	·00039422	·00026052	·00016876	·00010713	·00006664	·00004061	·00002424	1·6
1·7	·00053489	·00036254	·00024090	·00015691	·00010016	·00006265⁻	·00003839	·00002305⁻	·00001355⁺	1·7
1·8	·00032970	·00022028	·00014426	·00009260	·00005824	·00003589	·00002166	·00001281	·00000742	1·8
1·9	·00019919	·00013116	·00008465⁺	·00005354	·00003317	·00002014	·00001197	·00000697	·00000398	1·9
2·0	·00011793	·00007653	·00004866	·00003032	·00001851	·00001107	·00000648	·0⁴03716	·0⁴02087	2·0
2·1	·00006841	·00004374	·00002740	·00001682	·00001011	·00000595⁺	·0⁴03434	·0⁴01939	·0⁴01072	2·1
2·2	·00003888	·00002449	·00001512	·00000914	·00000541	·0⁴03138	·0⁴01782	·0⁴00991	·0⁴00539	2·2
2·3	·00002164	·00001343	·00000817	·00000486	·0⁴02835⁺	·0⁴01619	·0⁴00905⁺	·0⁴00496	·0⁴00266	2·3
2·4	·00001180	·00000721	·00000432	·0⁴02533	·0⁴01454	·0⁴00818	·0⁴00450⁺	·0⁴00243	·0⁴001280	2·4
2·5	·00000630	·00000379	·0⁴02237	·0⁴01292	·0⁴00730	·0⁴00404	·0⁴00219	·0⁴001163	·0⁴000604	2·5
2·6	·00000330	·0⁴01954	·0⁴01135⁻	·0⁴00645⁺	·0⁴00359	·0⁴00196	·0⁴001044	·0⁴000545⁺	·0⁴000279	2·6

0^4 indicates that four zeros must be placed before the figures that follow.

k	d/N for $r = -\cdot75$									k
	$h = 0{\cdot}9$	$h = 1{\cdot}0$	$h = 1{\cdot}1$	$h = 1{\cdot}2$	$h = 1{\cdot}3$	$h = 1{\cdot}4$	$h = 1{\cdot}5$	$h = 1{\cdot}6$	$h = 1{\cdot}7$	
0·0	·01273898	·00915625$^+$	·00646129	·00447528	·00304167	·00202812	·00132639	·00085066	·00053489	0·0
0·1	·00962023	·00682714	·00475554	·00325055$^+$	·00217976	·00143371	·00092475$^-$	·00058481	·00036254	0·1
0·2	·00713469	·00499785$^+$	·00343554	·00231690	·00153259	·00099416	·00063230	·00039422	·00024090	0·2
0·3	·00519483	·00359109	·00243550$^+$	·00162018	·00105695$^+$	·00067606	·00042391	·00026052	·00015691	0·3
0·4	·00371236	·00253193	·00169384	·00111127	·00071483	·00045077	·00027861	·00016876	·00010016	0·4
0·5	·00260314	·00175128	·00115543	·00074745$^+$	·00047401	·00029464	·00017948	·00010713	·00006265$^-$	0·5
0·6	·00179065$^-$	·00118805$^+$	·00077289	·00049291	·00030812	·00018876	·00011331	·00006664	·00003839	0·6
0·7	·00120807	·00079032	·00050687	·00031864	·00019631	·00011851	·00007009	·00004061	·00002305$^-$	0·7
0·8	·00079919	·00051544	·00032585$^-$	·00020188	·00012256	·00007290	·00004248	·00002424	·00001355$^+$	0·8
0·9	·00051832	·00032951	·00020530	·00012534	·00007497	·00004393	·00002522	·00001418	·00000780	0·9
1·0	·00032951	·00020645$^+$	·00012675$^+$	·00007625$^-$	·00004493	·00002594	·00001466	·00000812	0^{4}44401	1·0
1·1	·00020530	·00012675$^+$	·00007668	·00004544	·00002638	·00001500$^-$	·00000835$^-$	0^{4}04552	0^{4}02430	1·1
1·2	·00012534	·00007625$^-$	·00004544	·00002653	·00001517	·00000849	0^{4}04656	0^{4}02500$^-$	0^{4}01314	1·2
1·3	·00007497	·00004493	·00002638	·00001517	·00000854	0^{4}04709	0^{4}02542	0^{4}01344	0^{4}00695$^+$	1·3
1·4	·00004393	·00002594	·00001500$^-$	·00000849	0^{4}04709	0^{4}02557	0^{4}01359	0^{4}00707	0^{4}00360	1·4
1·5	·00002522	·00001466	·00000835$^-$	0^{4}04656	0^{4}02542	0^{4}01359	0^{4}00711	0^{4}00364	0^{4}001827	1·5
1·6	·00001418	·00000812	0^{4}04552	0^{4}02500$^-$	0^{4}01344	0^{4}00707	0^{4}00364	0^{4}001838	0^{4}000907	1·6
1·7	·00000780	0^{4}04401	0^{4}02430	0^{4}01314	0^{4}00695$^+$	0^{4}00360	0^{4}001827	0^{4}000907	0^{4}000441	1·7
1·8	0^{4}04207	0^{4}02336	0^{4}01270	0^{4}00676	0^{4}00352	0^{4}001797	0^{4}000896	0^{4}000438	0^{4}000210	1·8
1·9	0^{4}02220	0^{4}01214	0^{4}00650$^-$	0^{4}00340	0^{4}001747	0^{4}000877	0^{4}000431	0^{4}000207	0^{4}000098	1·9
2·0	0^{4}01147	0^{4}00618	0^{4}00325$^+$	0^{4}001679	0^{4}000848	0^{4}000419	0^{4}000202	0^{4}000096	0^{4}000044	2·0
2·1	0^{4}00580	0^{4}00308	0^{4}001596	0^{4}000810	0^{4}000403	0^{4}000196	0^{4}000094	0^{4}000043	0^{4}000020	2·1
2·2	0^{4}00287	0^{4}001499	0^{4}000766	0^{4}000385$^+$	0^{4}000187	0^{4}000090	0^{4}000041	0^{4}000019	0^{4}000009	2·2
2·3	0^{4}001393	0^{4}000715$^+$	0^{4}000359	0^{4}000177	0^{4}000085$^+$	0^{4}000040	0^{4}000019	0^{4}000008	0^{4}000004	2·3
2·4	0^{4}000661	0^{4}000334	0^{4}000166	0^{4}000080	0^{4}000038	0^{4}000018	0^{4}000008	0^{4}000004	0^{4}000002	2·4
2·5	0^{4}000307	0^{4}000153	0^{4}000074	0^{4}000035$^+$	0^{4}000017	0^{4}000008	0^{4}000003	0^{4}000002	0^{4}000001	2·5
2·6	0^{4}000139	0^{4}000068	0^{4}000033	0^{4}000015$^+$	0^{4}000007	0^{4}000003	0^{4}000001	0^{4}000001		2·6

k	d/N for $r = -\cdot75$									k
	$h = 1{\cdot}8$	$h = 1{\cdot}9$	$h = 2{\cdot}0$	$h = 2{\cdot}1$	$h = 2{\cdot}2$	$h = 2{\cdot}3$	$h = 2{\cdot}4$	$h = 2{\cdot}5$	$h = 2{\cdot}6$	
0·0	·00032970	·00019919	·00011793	·00006841	·00003888	·00002164	·00001180	·00000630	·00000330	0·0
0·1	·00022028	·00013116	·00007653	·00004374	·00002449	·00001343	·00000721	·00000379	0^{4}01954	0·1
0·2	·00014426	·00008465$^+$	·00004866	·00002740	·00001512	·00000817	·00000432	0^{4}02237	0^{4}01135$^-$	0·2
0·3	·00009260	·00005354	·00003032	·00001682	·00000914	·00000486	0^{4}02533	0^{4}01292	0^{4}00645$^+$	0·3
0·4	·00005824	·00003317	·00001851	·00001011	·00000541	0^{4}02835$^+$	0^{4}01454	0^{4}00730	0^{4}00359	0·4
0·5	·00003589	·00002014	·00001107	·00000595$^+$	0^{4}03138	0^{4}01619	0^{4}00818	0^{4}00404	0^{4}00196	0·5
0·6	·00002166	·00001197	·00000648	0^{4}03434	0^{4}01782	0^{4}00905$^+$	0^{4}00450$^+$	0^{4}00219	0^{4}001044	0·6
0·7	·00001281	·00000697	0^{4}03716	0^{4}01939	0^{4}00991	0^{4}00496	0^{4}00243	0^{4}001163	0^{4}000545$^+$	0·7
0·8	·00000742	0^{4}03976	0^{4}02087	0^{4}01072	0^{4}00539	0^{4}00266	0^{4}001280	0^{4}000604	0^{4}000279	0·8
0·9	0^{4}04207	0^{4}02220	0^{4}01147	0^{4}00580	0^{4}00287	0^{4}001393	0^{4}000661	0^{4}000307	0^{4}000139	0·9
1·0	0^{4}02336	0^{4}01214	0^{4}00618	0^{4}00308	0^{4}001499	0^{4}000715$^+$	0^{4}000334	0^{4}000153	0^{4}000068	1·0
1·1	0^{4}01270	0^{4}00650$^-$	000325$^+$	0^{4}001596	0^{4}000766	0^{4}000359	0^{4}000166	0^{4}000074	0^{4}000033	1·1
1·2	0^{4}00676	0^{4}00340	0^{4}001679	0^{4}000810	0^{4}000385$^+$	0^{4}000177	0^{4}000080	0^{4}000035$^+$	0^{4}000015$^+$	1·2
1·3	0^{4}00352	0^{4}001747	0^{4}000848	0^{4}000403	0^{4}000187	0^{4}000085$^+$	0^{4}000038	0^{4}000017	0^{4}000007	1·3
1·4	0^{4}001797	0^{4}000877	0^{4}000419	0^{4}000196	0^{4}000090	0^{4}000040	0^{4}000018	0^{4}000008	0^{4}000003	1·4
1·5	0^{4}000896	0^{4}000431	0^{4}000202	0^{4}000094	0^{4}000041	0^{4}000019	0^{4}000008	0^{4}000003	0^{4}000001	1·5
1·6	0^{4}000438	0^{4}000207	0^{4}000096	0^{4}000043	0^{4}000019	0^{4}000008	0^{4}000004	0^{4}000002	0^{4}000001	1·6
1·7	0^{4}000210	0^{4}000098	0^{4}000044	0^{4}000020	0^{4}000009	0^{4}000004	0^{4}000002	0^{4}000001		1·7
1·8	0^{4}000098	0^{4}000045$^-$	0^{4}000020	0^{4}000009	0^{4}000004	0^{4}000002	0^{4}000001			1·8
1·9	0^{4}000045$^-$	0^{4}000020	0^{4}000009	0^{4}000004	0^{4}000002	0^{4}000001				1·9
2·0	0^{4}000020	0^{4}000009	0^{4}000004	0^{4}000002	0^{4}000001					2·0
2·1	0^{4}000009	0^{4}000004	0^{4}000002	0^{4}000001						2·1
2·2	0^{4}000004	0^{4}000002	0^{4}000001							2·2
2·3	0^{4}000002	0^{4}000001								2·3
2·4	0^{4}000001									2·4
2·5										2·5
2·6										2·6

0^4 indicates that four zeros must be placed before the figures that follow.

A THEORY OF THE SAMPLING DISTRIBUTION OF STANDARD DEVIATIONS.

By T. KONDO.

Section I. Moment Coefficients of the Standard Deviations obtained in Sampling in terms of those of Variance.

(1) *Introduction.*

The standard deviation is one of the most important statistical constants, and with regard to it numerous researches have been made.

Suppose that from an infinite population, in which a character is measured by a variate x, samples of size N are drawn randomly and that this process is repeated indefinitely many times, then the standard deviation σ of the variate x will vary from sample to sample. I propose to consider here the distribution of σ in such cases. Concerning this problem several researches have already been made*, but some of them are only for a particular, not a general, parent distribution, and in others the degree of approximation in the results is not close enough for many purposes. I want here to deduce some general formulae for the sampling distribution of σ to a degree of approximation higher than that already obtained from a new point of view and by a different method of deduction.

Now any distribution law of a variate x can be defined by the moment coefficients for this distribution. If we can find the first four moment coefficients or, in the usual notation, μ_1' about a fixed origin and μ_2, μ_3, μ_4 about the mean, then we have as a rule enough information to define the distribution of frequency with sufficient accuracy for practical purposes.

The deduction of formulae for μ_1' and μ_r $(r=2, 3, 4)$ of σ in sampling is the primary object of this paper.

(A) *The first Method of Deduction.*

(2) Let $\phi(x)$ be the probability function of a continuous variate x, λ_s the sth semi-invariant of the distribution of x, and μ_r' the rth moment coefficient about a fixed origin; then the λ's are defined by the identity† with respect to ω

$$e^{\sum_{r=1}^{\infty} \frac{\lambda_r \omega^r}{r!}} = \int_{-\infty}^{+\infty} \phi(x)\, e^{x\omega}\, dx \quad \text{.....(1).}$$

* See "Student," *Biometrika*, Vol. VI. (March, 1908); K. Pearson, *Ibid.* Vol. XII. p. 277 (Nov. 1918); C. C. Craig, *Metron*, Vol. VII. No. 4 (Dec. 1928).

† Originally due to Thiele.

Expanding the right-hand side, we have

$$\int_{-\infty}^{+\infty}\phi(x)\,e^{x\omega}\,dx=\sum_{i=0}^{\infty}\frac{\omega^i}{i!}\int_{-\infty}^{+\infty}\phi(x)\,x^i\,dx$$
$$=\sum_{i=0}^{\infty}\frac{\mu_i'}{i!}\,\omega^i.$$

Equating the coefficient of the same powers of ω, we get the following well-known equations between the λ's and μ''s,

$$\mu_1'=\lambda_1,\quad \mu_2'=\lambda_2+\lambda_1^2,$$
$$\mu_3'=\lambda_3+3\lambda_2\lambda_1+\lambda_1^3,$$
$$\mu_4'=\lambda_4+4\lambda_3\lambda_1+3\lambda_2^2+6\lambda_2\lambda_1^2+\lambda_1^4 \quad\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

and so on.

Now let us choose the origin at the mean of x; then since $\mu_1'=\lambda_1=0$, we have

$$\mu_2=\lambda_2,\quad \mu_3=\lambda_3,\quad \mu_4=\lambda_4+3\lambda_2^2,$$
$$\mu_5=\lambda_5+10\lambda_3\lambda_2,\quad \mu_6=\lambda_6+15\lambda_4\lambda_2+10\lambda_3^2+15\lambda_2^3,$$
$$\mu_7=\lambda_7+21\lambda_5\lambda_2+35\lambda_4\lambda_3+105\lambda_3\lambda_2^2,$$
$$\mu_8=\lambda_8+28\lambda_6\lambda_2+56\lambda_5\lambda_3+35\lambda_4^2+210\lambda_4\lambda_2^2+280\lambda_3^2\lambda_2+105\lambda_2^4,$$
$$\mu_9=\lambda_9+36\lambda_7\lambda_2+84\lambda_6\lambda_3+126\lambda_5\lambda_4+378\lambda_5\lambda_2^2+1260\lambda_4\lambda_3\lambda_2+280\lambda_3^3+1260\lambda_3\lambda_2^3$$
$$\ldots\ldots(3),$$

and so on.

(3) Let us consider the case where x is the standard deviation σ; then

$$e^{\sum_{r=1}^{\infty}\frac{\lambda_r\omega^r}{r!}}=\int_{-\infty}^{+\infty}\phi(\sigma)\,e^{\sigma\omega}\,d\sigma \quad\ldots\ldots\ldots\ldots\ldots\ldots(4).$$

But the frequency of σ is the same as that of the variance μ_2. Therefore, if $\bar{\mu}_2$ be the mean of μ_2 in samples, y the deviation of μ_2 from $\bar{\mu}_2$, and $\Phi(y)$ the frequency function of μ_2, then since

$$\phi(\sigma)\,d\sigma=\Phi(y)\,dy,$$

we have

$$e^{\sum_{r=1}^{\infty}\frac{\lambda_r\omega^r}{r!}}=\int_{-\infty}^{+\infty}\Phi(y)\,e^{\omega\sqrt{\bar{\mu}_2+y}}\,dy \quad\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

And

$$\int_{-\infty}^{+\infty}\Phi(y)\,e^{\omega\sqrt{\bar{\mu}_2+y}}\,dy=\int_{-\infty}^{+\infty}\Phi(y)\,e^{\omega\sqrt{\bar{\mu}_2}\left(1+\frac{y}{\bar{\mu}_2}\right)^{\frac{1}{2}}}dy$$
$$=\int_{-\infty}^{+\infty}\Phi(y)\left(\sum_{i=0}^{\infty}\frac{\omega^i(\bar{\mu}_2)^{i/2}\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}}{i!}\right)dy$$
$$=\sum_{i=0}^{\infty}\frac{\omega^i(\bar{\mu}_2)^{i/2}}{i!}\int_{-\infty}^{+\infty}\Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}dy.$$

Therefore if we write

$$a_i=\int_{-\infty}^{+\infty}\Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}dy \quad\ldots\ldots\ldots\ldots\ldots\ldots(6),$$

then

$$e^{\sum_{r=1}^{\infty}\frac{\lambda_r\omega^r}{r!}}=\sum_{i=0}^{\infty}\frac{a_i(\bar{\mu}_2)^{i/2}\,\omega^i}{i!} \quad\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

Differentiating the identity (7) with respect to ω and equating the coefficients of the same powers of ω, we have

$$a_1(\bar{\mu}_2)^{\frac{1}{2}} = \lambda_1, \quad a_2\bar{\mu}_2 = \lambda_2 + a_1\lambda_1(\bar{\mu}_2)^{\frac{1}{2}},$$
$$a_3(\bar{\mu}_2)^{\frac{3}{2}} = \lambda_3 + 2a_1\lambda_2(\bar{\mu}_2)^{\frac{1}{2}} + a_2\lambda_1\bar{\mu}_2,$$
$$a_4(\bar{\mu}_2)^2 = \lambda_4 + 3a_1\lambda_3(\bar{\mu}_2)^{\frac{1}{2}} + 3a_2\lambda_2\bar{\mu}_2 + a_3\lambda_1(\bar{\mu}_2)^{\frac{3}{2}},$$
$$\vdots \quad \text{.....(8).}$$

We shall assume that it is justifiable to insert into the integral (6) the expansion of the binomial*. Then

$$\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2} = 1 + c_{i.1}\left(\frac{y}{\bar{\mu}_2}\right) + c_{i.2}\left(\frac{y}{\bar{\mu}_2}\right)^2 + \ldots + c_{i.r}\left(\frac{y}{\bar{\mu}_2}\right)^r + \ldots,$$

where $c_{i.r}$ is the coefficient of the $(r+1)$th term of $(1+x)^{i/2}$, or numerically

$$c_{1.2} = \frac{1}{8}, \quad c_{1.3} = \frac{1}{16}, \quad c_{1.4} = -\frac{5}{128}, \quad c_{1.5} = \frac{7}{256},$$
$$c_{1.6} = -\frac{21}{1024}, \quad c_{1.7} = \frac{33}{2048}, \quad c_{1.8} = -\frac{429}{32768}, \quad \ldots;$$
$$c_{2.2} = c_{2.3} = \ldots = 0;$$
$$c_{3.2} = \frac{3}{8}, \quad c_{3.3} = -\frac{1}{16}, \quad c_{3.4} = \frac{3}{128}, \quad c_{3.5} = -\frac{3}{256},$$
$$c_{3.6} = \frac{7}{1024}, \quad c_{3.7} = -\frac{9}{2048}, \quad c_{3.8} = \frac{99}{32768}, \quad \ldots;$$
$$c_{4.2} = 1, \quad c_{4.3} = c_{4.4} = \ldots = 0; \quad \text{and so on} \quad \text{.....(9).}$$

Now let ${}_2M_p$ be the pth moment coefficient about the mean for the distribution of μ_2 due to random sampling, and let us write

$$m_p = \frac{{}_2M_p}{2^p\bar{\mu}_2{}^p} \quad \text{.....(10);}$$

then

$$a_i = \int_{-\infty}^{+\infty}\Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}dy$$
$$= 1 + \sum_{r=1}^{\infty}\frac{c_{i.r}}{(\bar{\mu}_2)^r}\int_{-\infty}^{+\infty}\Phi(y)\,y^r\,dy$$
$$= 1 + \sum_{r=1}^{\infty}\frac{c_{i.r}}{(\bar{\mu}_2)^r}\,{}_2M_r$$
$$= 1 + \sum_{r=1}^{\infty}2^r c_{i.r}m_r.$$

But ${}_2M_1 = 0$, consequently $m_1 = 0$. Therefore

$$a_i = 1 + \sum_{r=2}^{\infty}2^r c_{i.r}m_r \quad \text{.....(11)}$$
$$= 1 + 4c_{i.2}m_2 + 8c_{i.3}m_3 + 16c_{i.4}m_4 + \ldots \quad (i = 1, 2, 3, 4, \ldots).$$

* This condition is discussed later, see Art. (16).

If we eliminate the λ's from the equations (2) and (8), we get

$$\begin{aligned}
\mu_1' &= a_1\sqrt{\bar{\mu}_2}, \\
\mu_2' &= a_2\bar{\mu}_2, \\
\mu_3' &= a_3(\bar{\mu}_2)^{\frac{3}{2}}, \\
\mu_4' &= a_4(\bar{\mu}_2)^2 \qquad \text{.....(12),}
\end{aligned}$$

where the μ''s are the first four moment coefficients of σ about $\sigma=0$ in sampling.

Now, for simplicity, let us put

$$a_1=a, \quad a_3=a+a';$$

then from the equation (11)

$$\begin{aligned}
a &= 1-\frac{m_2}{2}+\frac{m_3}{2}-\frac{5}{8}m_4+\frac{7}{8}m_5-\frac{21}{16}m_6+\frac{33}{16}m_7-\frac{429}{128}m_8+\ldots, \\
a' &= 2m_2-m_3+m_4-\frac{5}{4}m_5+\frac{7}{4}m_6-\frac{21}{8}m_7+\frac{33}{8}m_8-\ldots \qquad \text{.....(13),}
\end{aligned}$$

and since $a_2=1$, $a_4=1+4m_2$, from the equation (11), the equations (12) become

$$\begin{aligned}
&\mu_1' = a\sqrt{\bar{\mu}_2}, \quad \mu_2' = \bar{\mu}_2, \\
&\mu_3' = (a+a')(\bar{\mu}_2)^{\frac{3}{2}}, \\
\text{and} \qquad &\mu_4' = (1+4m_2)(\bar{\mu}_2)^2 \qquad \text{.....(14).}
\end{aligned}$$

But if $\tilde{\sigma}$ is the standard deviation of the parent distribution and N is the size of repeated samples, it is well known that

$$\bar{\mu}_2 = \text{Mean } \mu_2 = \frac{N-1}{N}\tilde{\sigma}^2.$$

Therefore the above equations for μ''s become

$$\begin{aligned}
&\mu_1' = \sqrt{\frac{N-1}{N}}a\tilde{\sigma}, \quad \mu_2' = \frac{N-1}{N}\tilde{\sigma}^2, \\
&\mu_3' = \left(\frac{N-1}{N}\right)^{\frac{3}{2}}(a+a')\tilde{\sigma}^3, \\
\text{and} \qquad &\mu_4' = \left(\frac{N-1}{N}\right)^2(1+4m_2)\tilde{\sigma}^4 \qquad \text{.....(14').}
\end{aligned}$$

Now the mean of σ and the first three moment coefficients $\mu_r(\sigma)$ about the mean can easily be deduced from the equations (14) or (14') by the well-known equations connecting μ_r''s or μ_r's, and the following equations are the results obtained*:

$$\text{Mean } \sigma = \bar{\sigma} = a\sqrt{\bar{\mu}_2} = \sqrt{\frac{N-1}{N}}a\tilde{\sigma} \qquad \text{.....(15 } a\text{),}$$

$$\mu_2(\sigma) = (1-a^2)\bar{\mu}_2 = \left(\frac{N-1}{N}\right)(1-a^2)\tilde{\sigma}^2 \qquad \text{.....(15 } b\text{),}$$

* Cf. Craig, *loc. cit.*

$$\mu_3(\sigma) = [\alpha' - 2\alpha(1-\alpha^2)]\sqrt{\bar{\mu}_2{}^3}$$
$$= \left(\frac{N-1}{N}\right)^{\frac{3}{2}}[\alpha' - 2\alpha(1-\alpha^2)]\,\tilde{\sigma}^3 \quad \ldots\ldots\ldots\ldots(15\,c),$$

$$\mu_4(\sigma) = [4(m_2 - \alpha\alpha') + (1-\alpha^2)(1+3\alpha^2)]\,\bar{\mu}_2{}^2$$
$$= \left(\frac{N-1}{N}\right)^2[4(m_2 - \alpha\alpha') + (1-\alpha^2)(1+3\alpha^2)]\,\tilde{\sigma}^4 \ldots\ldots\ldots(15\,d),$$

and consequently

$$\sigma_\sigma = \text{S.D. of } \sigma$$
$$= \sqrt{(1-\alpha^2)\,\bar{\mu}_2} = \sqrt{(1-\alpha^2)\frac{N-1}{N}}\,\tilde{\sigma} \quad \ldots\ldots\ldots\ldots(16\,a),$$

$$\beta_1(\sigma) = \mu_3(\sigma)^2/\mu_2(\sigma)^3 = \frac{[\alpha' - 2\alpha(1-\alpha^2)]^2}{(1-\alpha^2)^3} \quad \ldots\ldots\ldots\ldots(16\,b),$$

$$\beta_2(\sigma) = \mu_4(\sigma)/\mu_2(\sigma)^2 = \frac{1+3\alpha^2}{1-\alpha^2} + 4\,\frac{m_2 - \alpha\alpha'}{(1-\alpha^2)^2} \quad \ldots\ldots\ldots\ldots(16\,c),$$

where $\bar{\mu}_2$ and ${}_2M_r$ are the mean of the variance μ_2 and the rth moment coefficient of μ_2 in sampling, further:

$$m_r = {}_2M_r/(2^r\bar{\mu}_2{}^r)$$

while α, α' are given by the equations (13).

The equations (15) and (16) give respectively the first four moment coefficients for the sampling distribution of σ, its standard deviation, also the β_1, and β_2 of σ in terms of m_2, α, α' and $\bar{\mu}_2$, or in terms of ${}_2M_r$. They will all be exact expressions provided that the expansion of $\left(1 + \frac{y}{\bar{\mu}_2}\right)^{r/2}$ within the integral of (6) is justified, a condition which is discussed later*.

(B) *The second Method of Deduction.*

(4) We can deduce the equation (15) and consequently (16) also from another point of view.

Since $$\sigma = \sqrt{\mu_2} = \sqrt{\bar{\mu}_2 + \delta\mu_2},$$

where $\delta\mu_2$ is the sampling deviation of μ_2 from its mean, if $\delta\sigma$ is the deviation of σ from $\sqrt{\bar{\mu}_2}$, not from the mean, then

$$\sigma = \sqrt{\bar{\mu}_2} + \delta\sigma = (\bar{\mu}_2)^{\frac{1}{2}}\sqrt{1 + \delta\mu_2/\bar{\mu}_2}\,;$$

therefore $$\delta\sigma = \sqrt{\bar{\mu}_2}\,\{(1+\Delta)^{\frac{1}{2}} - 1\} \quad \ldots\ldots\ldots\ldots(17),$$

where Δ stands for $\delta\mu_2/\bar{\mu}_2$.

Assuming $\left|\frac{\delta\mu_2}{\bar{\mu}_2}\right| < 1$ as before*, we have

$$\delta\sigma = \sqrt{\bar{\mu}_2}\,(a_1\Delta + a_2\Delta^2 + a_3\Delta^3 + \ldots),$$

where $a_r = c_{1,r}$ $(r = 1, 2, 3, \ldots)$ and is given by the equation (9) in Art. (3).

* See Art. (16).

Now let ν_r' be the rth moment coefficient of σ about $\sigma=\sqrt{\bar{\mu}_2}$, and let us use brackets [ ] for "mean in repeated samples"; then

$$\begin{aligned}\nu_1' &= \text{Mean}\,\delta\sigma = \sqrt{\bar{\mu}_2}\left(\sum_{r=1}^{\infty} c_{1,r}[\Delta^r]\right)\\ &= \sqrt{\bar{\mu}_2}\left(\sum_{r=1}^{\infty}\frac{c_{1,r}}{\bar{\mu}_2{}^r}[(\delta\mu_2)^r]\right)\\ &= \sqrt{\bar{\mu}_2}\left(\sum_{r=2}^{\infty} 2^r c_{1,r}\frac{{}^2M_r}{2^r\bar{\mu}_2{}^r}\right)\\ &= \sqrt{\bar{\mu}_2}\left(\sum_{r=2}^{\infty} 2^r c_{1,r} m_r\right),\end{aligned}$$

therefore $$\nu_1' = \sqrt{\bar{\mu}_2}(\alpha-1) \quad \text{.....(18 a).}$$

Also from the equation (17), if we find $(\delta\sigma)^r$ $(r=2, 3, 4)$, and write as follows:

$$(\delta\sigma)^2 = \bar{\mu}_2\{b_2\Delta^2 + b_3\Delta^3 + b_4\Delta^4 + \ldots\},$$
$$(\delta\sigma)^3 = (\bar{\mu}_2)^{\frac{3}{2}}\{c_3\Delta^3 + c_4\Delta^4 + c_5\Delta^5 + \ldots\},$$
$$(\delta\sigma)^4 = (\bar{\mu}_2)^2\{d_4\Delta^4 + d_5\Delta^5 + d_6\Delta^6 + \ldots\},$$

then, after calculation, we get the following values of the coefficients:

$$b_2=\frac{1}{4},\quad b_3=-\frac{1}{8},\quad b_4=\frac{5}{64},\quad b_5=-\frac{7}{128},$$
$$b_6=\frac{21}{512},\quad b_7=-\frac{33}{1024},\quad b_8=\frac{429}{16384},\quad \ldots;$$
$$c_3=\frac{1}{8},\quad c_4=-\frac{3}{32},\quad c_5=\frac{9}{128},\quad c_6=-\frac{7}{128},\quad c_7=\frac{45}{1024},\quad c_8=-\frac{297}{8192},\quad \ldots;$$
$$d_4=\frac{1}{16},\quad d_5=-\frac{1}{16},\quad d_6=\frac{7}{128},\quad d_7=-\frac{3}{64},\quad d_8=\frac{165}{4096},\quad \ldots.$$

Now if we find the mean values of $(\delta\sigma)^2$, $(\delta\sigma)^3$ and $(\delta\sigma)^4$, we get

$$\begin{aligned}\nu_2' &= \text{Mean}\,(\delta\sigma)^2 = \bar{\mu}_2\sum_{r=2}^{\infty}(b_r[\Delta^r])\\ &= \bar{\mu}_2\left(\sum_{r=2}^{\infty} 2^r b_r m_r\right) \quad \text{.....(18 b).}\end{aligned}$$

Similarly $$\nu_3' = \text{Mean}\,(\delta\sigma)^3 = (\bar{\mu}_2)^{\frac{3}{2}}\left(\sum_{r=3}^{\infty} 2^r c_r m_r\right) \quad \text{.....(18 c).}$$

$$\nu_4' = \text{Mean}\,(\delta\sigma)^4 = (\bar{\mu}_2)^2\left(\sum_{r=4}^{\infty} 2^r d_r m_r\right) \quad \text{.....(18 d).}$$

But

$$\begin{aligned}\nu_2' &= \bar{\mu}_2\left\{m_2 - m_3 + \frac{5}{4}m_4 - \frac{7}{4}m_5 + \frac{21}{8}m_6 - \frac{33}{8}m_7 + \frac{429}{64}m_8 - \ldots\right\}\\ &= \bar{\mu}_2\left\{2 - 2\left(1 - \frac{m_2}{2} + \frac{m_3}{2} - \frac{5}{8}m_4 + \frac{7}{8}m_5 - \frac{21}{16}m_6 + \frac{33}{16}m_7 - \frac{429}{128}m_8 + \ldots\right)\right\}\\ &= \bar{\mu}_2(2-2\sigma) \quad \text{.....(18 b').}\end{aligned}$$

Similarly we can transform the equations (18 *c*) and (18 *d*) into the following forms:

$$\nu_3' = (\overline{\mu}_2)^{\frac{3}{2}} \{\alpha' + 4(\alpha - 1)\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (18\,c'),$$

$$\nu_4' = (\overline{\mu}_2)^2 \{4m_2 + 8(1-\alpha) - 4\alpha'\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (18\,d').$$

Hence we can find the mean σ and the first three moment coefficients about the mean in the sampling distribution of σ.

In fact from the equations (18 *a*), (18 *b'*), (18 *c'*), (18 *d'*) and well-known formulae connecting ν''s and μ's, we get

$$\text{Mean } \sigma = (\overline{\mu}_2)^{\frac{1}{2}} + \nu_1' = \alpha\sqrt{\overline{\mu}_2},$$

$$\mu_2(\sigma) = \nu_2' - \nu_1'^2 = \overline{\mu}_2(1 - \alpha^2);$$

similarly

$$\mu_3(\sigma) = (\sqrt{\overline{\mu}_2})^{\frac{3}{2}} \{\alpha' + 4(\alpha - 1) - 3(\alpha - 1)(2 - 2\alpha) + 2(\alpha - 1)^3\}$$

$$= (\sqrt{\overline{\mu}_2})^{\frac{3}{2}} \{\alpha' - 2\alpha(1 - \alpha^2)\},$$

and

$$\mu_4(\sigma) = (\overline{\mu}_2)^2 \{4m_2 + 8(1-\alpha) - 4\alpha' - 4(\alpha - 1)[\alpha' - 2\alpha(1 - \alpha^2)] - 6(\alpha - 1)^2(1 - \alpha^2) - (\alpha - 1)^4\}$$

$$= (\overline{\mu}_2)^2 \{4(m_2 - \alpha\alpha') + (1 - \alpha^2)(1 + 3\alpha^2)\}.$$

These equations are the same as (15 *a*), (15 *b*), (15 *c*) and (15 *d*) respectively, already obtained in Art. (3).

Section II. Moment Coefficients of σ in terms of the Constants of the Sampled Population.

(5) Now if repeated random samples of size N be drawn from an infinite population which is specified by the standard deviation $\tilde{\sigma}$, and the constants $\tilde{\beta}_1, \tilde{\beta}_2, \ldots \tilde{\beta}_r, \ldots$, where

$$\beta_{2r-2} = \frac{\mu_{2r}}{(\mu_2)^r}, \quad \beta_{2r-1} = \frac{\mu_{2r+1}\mu_3}{(\mu_2)^{r+2}},$$

then it is well known* that the first four moment coefficients of the sampling distribution of the variance are given by

$${}_2M_1' = \text{Mean } \mu_2 = \frac{N-1}{N}\tilde{\sigma}^2,$$

$${}_2M_2 = \frac{(N-1)^2}{N^3}\left(\tilde{\beta}_2 - 3 + \frac{2N}{N-1}\right)\tilde{\sigma}^4,$$

$${}_2M_3 = \frac{1}{N^2}\left(k_{32} - \frac{k_{33}}{N} + \frac{k_{34}}{N^2} - \frac{k_{35}}{N^3}\right)\tilde{\sigma}^6,$$

$${}_2M_4 = \frac{1}{N^2}\left(k_{42} + \frac{k_{43}}{N} - \frac{k_{44}}{N^2} + \frac{k_{45}}{N^3} - \frac{k_{46}}{N^4} + \frac{k_{47}}{N^5}\right)\tilde{\sigma}^8 \quad \ldots\ldots\ldots\ldots (19),$$

* See A. A. Tchouproff, *Biometrika*, Vol. XII. pp. 193—4; A. E. R. Church, *Ibid.* Vol. XVII. pp. 79—83.

where $k_{32}=\tilde{\beta}_4-3\tilde{\beta}_2-6\tilde{\beta}_1+2$, $k_{33}=3\tilde{\beta}_4-21\tilde{\beta}_2-18\tilde{\beta}_1+26$

$k_{34}=3\tilde{\beta}_4-33\tilde{\beta}_2-22\tilde{\beta}_1+54$, $k_{35}=\tilde{\beta}_4-15\tilde{\beta}_2-10\tilde{\beta}_1+30$;

$k_{42}=3(\tilde{\beta}_2-1)^2$,

$k_{43}=\tilde{\beta}_6-4\tilde{\beta}_4-24\tilde{\beta}_3-15\tilde{\beta}_2^2+48\tilde{\beta}_2+96\tilde{\beta}_1-30$,

$k_{44}=4\tilde{\beta}_6-40\tilde{\beta}_4-96\tilde{\beta}_3-54\tilde{\beta}_2^2+336\tilde{\beta}_2+528\tilde{\beta}_1-306$,

$k_{45}=6\tilde{\beta}_6-96\tilde{\beta}_4-176\tilde{\beta}_3-102\tilde{\beta}_2^3+924\tilde{\beta}_2+1232\tilde{\beta}_1-1044$,

$k_{46}=4\tilde{\beta}_6-88\tilde{\beta}_4-160\tilde{\beta}_3-95\tilde{\beta}_2^2+1050\tilde{\beta}_2+1360\tilde{\beta}_1-1395$,

$k_{47}=\tilde{\beta}_6-28\tilde{\beta}_4-56\tilde{\beta}_3-35\tilde{\beta}_2^2+420\tilde{\beta}_2+560\tilde{\beta}_1-630$.

Therefore if the approximation is good enough up to ${}_2M_4$, we can express μ_1' and μ_2, μ_3 and μ_4 at once in terms of $\tilde{\sigma}$ and the $\tilde{\beta}$'s, i.e. in terms of the constants of the parent distribution.

Suppose further that we neglect terms of higher order than N^{-2}, then

$$a=1-\frac{m_2}{2}+\frac{m_3}{2}-\frac{5}{8}m_4,\quad a'=2m_2-m_3+m_4,$$

where
$$m_2=\frac{{}_2M_2}{4\bar{\mu}_2^2}=\frac{1}{4N}\left\{\tilde{\beta}_2-3+\frac{2N}{N-1}\right\}$$
$$=\frac{1}{2N}\left(\frac{\tilde{\beta}_2-1}{2}+\frac{1}{2N}\right),$$
$$m_3=\frac{1}{8N^2}\{\tilde{\beta}_4-3\tilde{\beta}_2-6\tilde{\beta}_1+2\} \quad \text{.....(20)},$$
$$m_4=\frac{3(\tilde{\beta}_2-1)^2}{16N^2} \text{ and } \bar{\mu}_2=\frac{N-1}{N}\tilde{\sigma}^2.$$

Substituting these values of $a, a', \bar{\mu}_2$ and the m's into the general and fundamental equations (15) and (16), after transformation and simplification, we get

$$\text{Mean } \sigma=\tilde{\sigma}\left\{1-\frac{\tilde{\beta}_2+3}{8N}+\frac{8\tilde{\beta}_4-15\tilde{\beta}_2^2+14\tilde{\beta}_2-48\tilde{\beta}_1-55}{128N^2}\right\} \text{.....(21 } a),$$

$$\mu_2(\sigma)=\frac{\tilde{\sigma}^2}{4N}\left\{\tilde{\beta}_2-1-\frac{4\tilde{\beta}_4-7\tilde{\beta}_2^2+10\tilde{\beta}_2-24\tilde{\beta}_1-23}{8N}\right\} \text{.....(21 } b),$$

$$\mu_2(\sigma)=\frac{\tilde{\sigma}^3}{16N^2}\{2\tilde{\beta}_4-3\tilde{\beta}_2^2-12\tilde{\beta}_1+1\} \text{.....(21 } c),$$

and
$$\mu_4(\sigma)=\frac{3}{16N^2}(\tilde{\beta}_2-1)^2\tilde{\sigma}^4 \text{.....(21 } d);$$

consequently

$$\sigma_\sigma=\frac{1}{2}\sqrt{\frac{\tilde{\beta}_2-1}{N}}\left\{1-\frac{4\tilde{\beta}_4-7\tilde{\beta}_2^2+10\tilde{\beta}_2-24\tilde{\beta}_1-23}{16(\tilde{\beta}_2-1)N}\right\}\tilde{\sigma} \text{ .....(21 } e).$$

In the cases where the parent distribution is normal; since

$$\tilde{\beta}_1 = 0, \quad \tilde{\beta}_2 = 3 \quad \text{and} \quad \tilde{\beta}_4 = 15,$$

we have

$$\text{Mean } \sigma = \left(1 - \frac{3}{4N} - \frac{7}{32N^2}\right)\tilde{\sigma},$$

$$\mu_2(\sigma) = \frac{1}{2N}\left(1 - \frac{1}{4N}\right)\tilde{\sigma}^2,$$

$$\mu_3(\sigma) = \frac{1}{4N^2}\tilde{\sigma}^3, \quad \mu_4(\sigma) = \frac{3}{4N^2}\tilde{\sigma}^4 \quad \text{.....(22 } a\text{)},$$

and

$$\sigma_\sigma = \frac{1}{\sqrt{2N}}\left(1 - \frac{1}{8N}\right)\tilde{\sigma},$$

$$\beta_1(\sigma) = \frac{1}{2N}, \quad \beta_2(\sigma) = 3\left\{1 + \frac{1}{2N}\right\} \quad \text{.....(22 } b\text{)}.$$

Some of the equations (21) and (22) have been already deduced by Prof. K. Pearson*.

Now we must notice that the above approximate formulae have been obtained by neglecting terms of the general equations (15) and (16) in two different ways. Firstly we neglected the moment coefficients ${}_2M_r$ for $r > 4$, and secondly we neglected those terms of order higher than N^{-2}.

But such a double method of approximation cannot be carried through correctly unless we know the order of ${}_2M_r$.

(6) Now if λ_r be the semi-invariant for the sampling distribution of the variance μ_2, and N the size of the repeated random samples, in the semi-invariant theory, it is known that (i) $\lambda_r\,(r = 2, 3, 4, \ldots)$ is independent of the origin, and (ii) $\lambda_r\,(r = 2, 3, 4, \ldots)$ is of order $r - 1$ in N^{-1}, and from the general equations (3), we have

$$\begin{aligned}
&{}_2M_2 = \lambda_2, && {}_2M_3 = \lambda_3,\\
&{}_2M_4 = \lambda_4 + 3\lambda_2^2, && {}_2M_5 = \lambda_5 + 10\lambda_3\lambda_2,\\
&{}_2M_6 = \lambda_6 + 15\lambda_4\lambda_2 + 10\lambda_3^2 + 15\lambda_2^3, && \text{and so on} \quad \text{.....(23)}.
\end{aligned}$$

Thus we can find the order of the coefficient ${}_2M_r$.

For instance, if we can assume that the approximation is good enough only up to the order N^{-2}, since ${}_2M_r\,(r \geqq 5)$ is of order N^{-3} or higher, we can neglect these ${}_2M_r$ and at the same time we can neglect those terms of $m_r\,(r = 2, 3, 4)$ of order higher than N^{-2}.

This is the case treated in the foregoing article as an example.

Secondly, if we can assume that the approximation is good enough when we include terms up to the order N^{-3} only, we may neglect ${}_2M_r\,(r = 7, 8, 9, 10, \ldots)$, and at the same time those terms of $m_r\,(r \leqslant 6)$ of order higher than N^{-3}.

* See *Biometrika*, Vol. XII. p. 277 (Nov. 1918). See also Craig, *loc. cit.*

This is the case treated by Dr C. C. Craig. His results are given as the semi-invariants of the sampling distribution of σ in terms of those of the parent distribution, but his final results correspond to those which will be obtained by neglecting the highest order terms in N^{-1} of my general approximate formulae, given in Art. (9).

The formulae, obtained in these two cases, give us good estimates of the mean σ, $\mu_2(\sigma)$ and σ_σ if N be not very small, and also of $\mu_3(\sigma)$ and $\beta_1(\sigma)$ if N be large, but we cannot get good estimates of $\mu_4(\sigma)$ and $\beta_2(\sigma)$ unless N be very large.

Thirdly, if it be necessary to proceed to a further approximation and include terms up to the order N^{-4}, we can neglect ${}_2M_r$ $(r \geqq 9)$, for they are of order N^{-5} or higher, and at the same time we can neglect those terms of m_r $(r = 2, 3, \ldots, 8)$ of order N^{-5} or higher. Such an approximation is necessary for the calculation of $\beta_2(\sigma)$, and with such approximations I shall now deal.

In this case we have

$$
\begin{aligned}
&{}_2M_2 = \lambda_2, \quad {}_2M_3 = \lambda_3, \quad {}_2M_4 = \lambda_4 + 3\lambda_2^2,\\
&{}_2M_5 = \lambda_5 + 10\lambda_3\lambda_2,\\
&{}_2M_6 = 15\lambda_4\lambda_2 + 10\lambda_3^2 + 15\lambda_2^3 \text{ (approximately)},\\
&{}_2M_7 = 105\lambda_2^2\lambda_3 \qquad (\quad ,, \quad),\\
\text{and} \quad &{}_2M_8 = 105\lambda_2^4 \qquad (\quad ,, \quad) \ldots\ldots\ldots\ldots (24).
\end{aligned}
$$

And also
$$\alpha = 1 - \frac{m_2}{2} + \frac{m_3}{2} - \frac{5}{8}m_4 + \frac{7}{8}m_5 - \frac{21}{16}m_6 + \frac{33}{16}m_7 - \frac{429}{128}m_8 \quad \ldots\ldots (25\,a),$$

$$\alpha' = 2m_2 - m_3 + m_4 - \frac{5}{4}m_5 + \frac{7}{4}m_6 - \frac{21}{8}m_7 + \frac{33}{8}m_8 \quad \ldots\ldots\ldots\ldots\ldots (25\,b).$$

Now the equation (24) suggests that to get this degree of approximation we have at first to find ${}_2M_5$ or the corresponding semi-invariant λ_5.

The values of ${}_2M_5$ or λ_5 for the sampling distribution of μ_2 were not known until recently, but they have now been found by R. A. Fisher and published with many other valuable results in an important paper*.

(7) R. A. Fisher's "cumulative moment function" κ_r is the same as the usual semi-invariant λ_r, introduced by Thiele, but his "k-function" is a new function defined as follows:

$$
\begin{aligned}
k_1 &= \frac{1}{N}s_1 = \text{Mean } x,\\
k_2 &= \frac{N}{N-1}\mu_2 = \frac{1}{N-1}\left(s_2 - \frac{s_1^2}{N}\right),\\
k_3 &= \frac{N^2}{(N-1)(N-2)}\mu_3\\
&= \frac{N^2}{(N-1)(N-2)}\left(s_2 - \frac{3s_2s_1}{N} + \frac{2s_1^3}{N^2}\right),\\
&\quad \text{and so on,}
\end{aligned}
$$

* R. A. Fisher, *Proceedings of the London Mathematical Society*, Ser. 2, Vol. xxx, Part 3 (Dec. 1928).

where $s_r = S(x^r)$ is the power sum of order r for any variate x, and

$$\mu_r = \frac{1}{N} S\,(x - \text{Mean } x)^r$$

as usual.

Fisher uses $\kappa(2^5)$ to denote the fifth cumulative moment function of k_2, due to sampling, and his expression for $\kappa(2^5)$ is as follows:

$$\begin{aligned}\kappa(2^5) = {} & \frac{\kappa_{10}}{N^4} + \frac{40\kappa_8\kappa_2}{N^3(N-1)} + \frac{80(N-2)\kappa_7\kappa_3}{N^3(N-1)^2} + \frac{40(5N^2-12N+9)\kappa_6\kappa_4}{N^3(N-1)^3} \\ & + \frac{16(N-2)(6N^2-12N+17)\kappa_5^2}{N^3(N-1)^4} + \frac{480\kappa_6\kappa_2^2}{N^2(N-1)^2} + \frac{1280(N-2)\kappa_5\kappa_3\kappa_2}{N^2(N-1)^3} \\ & + \frac{320(4N^2-9N+6)\kappa_4^2\kappa_2}{N^2(N-1)^4} + \frac{480(2N^2-7N+6)\kappa_4\kappa_3^2}{N^2(N-1)^4} + \frac{1920\kappa_4\kappa_2^3}{N(N-1)^3} \\ & + \frac{1920(N-2)\kappa_3^3\kappa_2^2}{N(N-1)^4} + \frac{384\kappa_2^5}{(N-1)^4},\end{aligned}$$

where κ_s is the sth cumulative function of the parent distribution. In this equation if we neglect terms of order N^{-5} or higher, we have

$$\begin{aligned}\kappa(2^5) &= \frac{1}{N^4}\{\kappa_{10} + 40\kappa_8\kappa_2 + 80\kappa_7\kappa_3 + 200\kappa_6\kappa_4 + 96\kappa_5^2 + 480\kappa_6\kappa_2^2 + 1280\kappa_5\kappa_3\kappa_2 \\ &\qquad + 1280\kappa_4^2\kappa_2 + 960\kappa_4\kappa_3^2 + 1920\kappa_4\kappa_2^3 + 1920\kappa_3^2\kappa_2^2 + 384\kappa_2^5\} \\ &= \frac{4}{N^4}\tau_0\tilde{\sigma}^{10}, \text{ say} \quad \ldots\ldots\ldots\

Therefore $$ {}_2M_5 = 10 \cdot {}_2M_2 \cdot {}_2M_3 + \frac{(N-1)^5}{N^5}\kappa(2^5) $$

$$ = 10 \cdot {}_2M_2 \cdot {}_2M_3 + \frac{4\tau_0}{N^4}\tilde{\sigma}^{10} \text{ (approximately)} \quad \text{.....(28).} $$

Now we can find the expressions for m's in terms of $\tilde{\beta}$'s which are necessary for our present purposes, and consequently can find the expressions for the mean σ and $\mu_r(\sigma)$ $(r=2, 3, 4)$ up to the order N^{-4}.

(9) Let us next proceed to find expressions for all the m's up to the order N^{-4}, which are necessary for our present purposes.

Since $$ {}_2M_2 = \frac{(N-1)^2}{N^3}\left(\tilde{\beta}_2 - 3 + \frac{2N}{N-1}\right)\tilde{\sigma}^4, \quad \bar{\mu}_2 = \frac{N-1}{N}\tilde{\sigma}^2 \text{.....(29),} $$

$$ m_2 = \frac{{}_2M_2}{4\bar{\mu}_2^2} = \frac{1}{2N}\left(\frac{\tilde{\beta}_2 - 1}{2} + \frac{1}{N-1}\right) $$

$$ = \frac{1}{2N}\left(p + \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^3}\right) \text{ (approximately) ...(30 } a\text{),} $$

where $$ p = \tfrac{1}{2}(\tilde{\beta}_2 - 1) \text{ .....(31).} $$

Similarly from the equations (24) we have

$$ m_3 = \frac{{}_2M_3}{8\bar{\mu}_2^3} = \frac{1}{8N^2}\left\{q - 6p + \frac{24p-8}{N} + \frac{4(6p-\tilde{\beta}_1)}{N^2}\right\} \text{ (approximately) (30 } b\text{),} $$

where $$ q = \tilde{\beta}_4 - 6\tilde{\beta}_1 - 1 \text{ .....(32),} $$

$$ m_4 = \frac{{}_2M_4}{16\bar{\mu}_2^4} = \frac{3}{4N^2}\left\{p^2 + \frac{s_3}{N} + \frac{2q + 8p^2 - 26p + 5}{N^2}\right\} \text{ (approximately) (30 } c\text{),} $$

where $$ s_3 = \tfrac{1}{12}\{\tilde{\beta}_6 - 24\tilde{\beta}_3 + 72\tilde{\beta}_1 - 4q - 12p^2 + 36p - 1\}, $$

$$ m_5 = \frac{{}_2M_5}{32\bar{\mu}_2^5} = \frac{\lambda_5}{32\bar{\mu}_2^5} + 10\,\frac{{}_2M_2 \cdot {}_2M_3}{32\bar{\mu}_2^5} $$

$$ = \frac{1}{8N^3}\left\{5p(q-6p) + \frac{1}{N}(120p^2 - 70p + 5q + \tau_0\right\} \text{ (approximately)} $$

$$ \text{.....(30 } d\text{),} $$

$$ m_6 = \frac{5}{8N^3}\left\{3p^3 + \frac{1}{4N}(36ps_3 + q^2 - 12pq)\right\} \text{ (approximately) .....(30 } e\text{),} $$

$$ m_7 = \frac{105p^2(q-6p)}{32N^4} \text{ (approximately) .....(30 } f\text{),} $$

and $$ m_8 = \frac{105p^4}{16N^4} \text{ (approximately) .....(30 } g\text{).} $$

If we substitute these values of the m's into the expressions (25 a) for α, and (25 b) for α', after simplification, we have

$$ \alpha = 1 - \frac{p}{4N} - \frac{\tau_2}{32N^2} - \frac{\tau_3}{128N^3} - \frac{\tau_4}{2048N^4} \text{ .....(33 } a\text{),} $$

and $$ \alpha' = \frac{1}{N}\left(p + \frac{\tau_2'}{8N} + \frac{\tau_3'}{32N^2} + \frac{\tau_4'}{256N^3}\right) \text{ .....(33 } b\text{),} $$

where

$$\tau_2 = 15p^2 + 12p - 2q + 8$$
$$= \tfrac{1}{4}(-8\tilde{\beta}_4 + 15\tilde{\beta}_2^2 - 6\tilde{\beta}_2 + 48\tilde{\beta}_1 + 31),$$
$$\tau_3 = 60s_3 + 315p^3 - 70p(q-6p) - 64(3p-1) + 32$$
$$= \tfrac{1}{8}(40\tilde{\beta}_6 - 280\tilde{\beta}_4\tilde{\beta}_2 + 120\tilde{\beta}_4 - 960\tilde{\beta}_3 + 1680\tilde{\beta}_2\tilde{\beta}_1 + 315\tilde{\beta}_2^3 - 225\tilde{\beta}_2^2 - 263\tilde{\beta}_2 + 2160\tilde{\beta}_1 + 1061),$$
$$\tau_4 = 1260p(\tilde{\beta}_6 - 24\tilde{\beta}_3 + 72\tilde{\beta}_1) - 224\tau_0 + 45045p^4 - 13860p^2q + 68040p^3 + 26160p^2 - 10080pq + 420q^2 - 13612p + 800q + 512\tilde{\beta}_1 + 5312,$$
$$= \tfrac{1}{16}\{-896(\tilde{\beta}_8 - 40\tilde{\beta}_5) + 1120\tilde{\beta}_6(9\tilde{\beta}_2 - 5) + 26880\tilde{\beta}_3(\tilde{\beta}_3/\tilde{\beta}_1 - 9\tilde{\beta}_2 - 7) + 45045\tilde{\beta}_2^4 - 44100\tilde{\beta}_2^3 + 10080\tilde{\beta}_1(33\tilde{\beta}_2^2 + 22\tilde{\beta}_2 + 24) - 4770\tilde{\beta}_2^2 - 66596\tilde{\beta}_2 + 532832\tilde{\beta}_1 - 38827\} \quad \ldots\ldots(34\,a);$$

and

$$\tau_2' = 6p^2 + 6p - q + 8$$
$$= \tfrac{1}{2}(3\tilde{\beta}_2^2 - 2\tilde{\beta}_4 + 12\tilde{\beta}_1 + 15),$$
$$\tau_3' = 2\tilde{\beta}_6 - 48\tilde{\beta}_3 + 144\tilde{\beta}_1 + 105p^3 + 126p^2 - 25pq - 24p - 8q + 62$$
$$= \tfrac{1}{8}(16\tilde{\beta}_6 - 384\tilde{\beta}_3 + 36\tilde{\beta}_4 - 100\tilde{\beta}_4\tilde{\beta}_2 + 105\tilde{\beta}_2^3 - 63\tilde{\beta}_2^2 + 600\tilde{\beta}_2\tilde{\beta}_1 - 185\tilde{\beta}_2 + 936\tilde{\beta}_1 + 70),$$
$$\tau_4' = 210p(\tilde{\beta}_6 - 24\tilde{\beta}_3 + 72\tilde{\beta}_1) - 40\tau_0 + 6930p^4 + 10710p^3 - 2205p^2q + 4296p^2 - 1680pq + 70q^2 - 3170p + 184q + 128\tilde{\beta}_1 + 1216$$
$$= \tfrac{1}{8}\{-80(\tilde{\beta}_8 - 40\tilde{\beta}_5) + 40\tilde{\beta}_6(21\tilde{\beta}_2 - 11) + 2\tilde{\beta}_4(280\tilde{\beta}_4 - 2205\tilde{\beta}_2^2 + 1450\tilde{\beta}_2 - 3360\tilde{\beta}_1 + 531) + 480\tilde{\beta}_3(5\tilde{\beta}_3/\tilde{\beta}_1 - 42\tilde{\beta}_2 - 38) + 180\tilde{\beta}_1(147\tilde{\beta}_2^2 + 106\tilde{\beta}_2 + 112\tilde{\beta}_1) + 3465\tilde{\beta}_2^4 - 3150\tilde{\beta}_2^3 - 738\tilde{\beta}_2^2 - 8894\tilde{\beta}_2 + 49372\tilde{\beta}_1 + 18613\} \quad \ldots\ldots(34\,b).$$

Substituting these values of α, α' and m_2 into the general equations for the Mean σ, $\mu_2(\sigma)$, $\mu_3(\sigma)$ and $\mu_4(\sigma)$, and neglecting terms of order N^{-5} or higher, after long transformations and simplifications I have obtained the following results:

$$\bar{\sigma} = \left\{1 - \frac{p+2}{4N} - \frac{\tau_2 - 4p + 4}{32N^2} - \frac{\tau_3 - 2\tau_2 - 4p + 8}{128N^3} - \frac{\tau_4 - 8(\tau_3 + \tau_2) - 32p + 80}{2048N^4}\right\}\tilde{\sigma} \quad \ldots\ldots(35\,a),$$

$$\mu_2(\sigma) = \frac{1}{2N}\left\{p + \frac{\tau_2 - p^2 - 8p}{8N} + \frac{\tau_3 - p\tau_2 - 4(\tau_2 - p^2)}{32N^2} + \frac{\tau_4 - 4(p+4)\tau_3 - \tau_2(\tau_2 - 16p)}{512N^3}\right\}\tilde{\sigma}^2 \quad \ldots\ldots(35\,b),$$

$$\mu_3(\sigma)=\frac{1}{8N^2}\left\{\tau_2'-\tau_2+3p^2+\frac{1}{4N}[\tau_3'-\tau_3+3p\tau_2-6(\tau_2'-\tau_2)-p^2(p+18)]\right.$$

$$+\frac{1}{64N^2}[2\tau_4'-\tau_4+12p\tau_3-24(\tau_3'-\tau_3)+3\tau_2^2-6p^2\tau_2+24(\tau_2'-\tau_2)$$

$$\left.-72p\tau_2+24p^2(p+3)]\right\}\tilde{\sigma}^3 \quad \text{.....(35 } c\text{)},$$

$$\mu_4(\sigma)=\frac{1}{4N^2}\left\{\tau_2-2\tau_2'+8+\frac{1}{4N}[\tau_3-2\tau_3'-2p(\tau_2-\tau_2')-8(\tau_2-2\tau_2')+3p^3-32]\right.$$

$$+\frac{1}{64N^2}[\tau_4-4\tau_4'-8p(\tau_3-\tau_3')-32(\tau_3-2\tau_3')+2\tau_2(2\tau_2'-2\tau_2+9p^2)$$

$$\left.+64p(\tau_2-\tau_2')+64(\tau_2-2\tau_2')-3p^3(p+32)]\right\}\tilde{\sigma}^4 \quad \text{.....(35 } d\text{)},$$

where $p=\frac{1}{2}(\tilde{\beta}_2-1)$, and the τ's, τ''s are given in (34 a) and (34 b).

To find σ_σ, $\beta_1(\sigma)$ and $\beta_2(\sigma)$, in particular numerical cases, it is most convenient to use the general equations (16) directly after the calculation of m_2, α and α'.

(10) Now let us consider the special cases where the approximation is adequate up to the orders N^{-2} and N^{-3}.

In the first case, neglecting terms of orders N^{-3} and N^{-4} in the general approximate equations (35), we get

$$\text{Mean } \sigma=\tilde{\sigma}\left(1-\frac{p+2}{4N}-\frac{\tau_2-4p+4}{32N^2}\right),$$

$$\mu_2(\sigma)=\frac{\tilde{\sigma}^2}{2N}\left(p+\frac{\tau_2-p^2-8p}{8N}\right),$$

$$\mu_3(\sigma)=\frac{\tilde{\sigma}^3}{8N^2}(\tau_2'-\tau_2+3p^2),$$

and

$$\mu_4(\sigma)=\frac{\tilde{\sigma}^4}{4N^2}(\tau_2-2\tau_2'+8) \quad \text{.....(36).}$$

But

$$-\tfrac{1}{4}(p+2)=-\tfrac{1}{8}(\tilde{\beta}_2+3),$$

$$\tau_2-4p+4=-\tfrac{1}{4}(8\tilde{\beta}_1-15\tilde{\beta}_2^2+14\tilde{\beta}_2-48\tilde{\beta}_1-55),$$

$$\tau_2-p^2-8p=-\tfrac{1}{2}(4\tilde{\beta}_4-7\tilde{\beta}_2^2+10\tilde{\beta}_2-24\tilde{\beta}_1-23),$$

$$\tau_2'-\tau_2+3p^2=\tfrac{1}{2}(2\tilde{\beta}_4-3\tilde{\beta}_2^2-12\tilde{\beta}_1+1),$$

and

$$\tau_2-2\tau_2'+8=\tfrac{3}{4}(\tilde{\beta}_2-1)^2 \quad \text{.....(37).}$$

If we substitute these expressions in the equations (36), we get a set of equations for the mean σ, $\mu_2(\sigma)$, $\mu_3(\sigma)$ and $\mu_4(\sigma)$ which are the same as the equations (21), as we should expect.

In the second case, neglecting terms of order N^{-4} in (35), we have

$$\text{Mean } \sigma = \tilde{\sigma}\left\{1 - \frac{p+2}{4N} - \frac{\tau_2 - 4p + 4}{32N^2} - \frac{\tau_3 - 2\tau_2 - 4p + 8}{128N^3}\right\},$$

$$\mu_2(\sigma) = \frac{\tilde{\sigma}^2}{2N}\left\{p + \frac{\tau_2 - p^2 - 8p}{8N} + \frac{\tau_3 - p\tau_2 - 4(\tau_2 - p^2)}{32N^2}\right\},$$

$$\mu_3(\sigma) = \frac{\tilde{\sigma}^3}{8N^2}\left\{\tau_2' - \tau_2 + 3p^2 + \frac{\tau_3' - \tau_3 + 3p\tau_2 - 6(\tau_2' - \tau_2) - p^2(p+18)}{4N}\right\},$$

$$\mu_4(\sigma) = \frac{\tilde{\sigma}^4}{4N^2}\left\{\tau_2 - 2\tau_2' + 8 + \frac{\tau_3 - 2\tau_3' - 2p(\tau_2 - \tau_2') - 8(\tau_2 - 2\tau_2') + 3p^3 - 32}{4N}\right\}$$

.....(38).

But

$$\tau_3 - 2\tau_2 - 4p + 8 = \tfrac{1}{8}\{40\tilde{\beta}_6 - 8\tilde{\beta}_4(35\tilde{\beta}_2 - 19) - 960\tilde{\beta}_3 + 315\tilde{\beta}_2^3 + 1680\tilde{\beta}_2\tilde{\beta}_1 - 285\tilde{\beta}_2^2 - 255\tilde{\beta}_2 + 1968\tilde{\beta}_1 + 1017\},$$

$$\tau_3 - p\tau_2 - 4(\tau_2 - p^2) = \tfrac{1}{2}\{10\tilde{\beta}_6 - 4\tilde{\beta}_4(17\tilde{\beta}_2 - 11) - 240\tilde{\beta}_3 + 75\tilde{\beta}_2^3 - 79\tilde{\beta}_2^2 - 67\tilde{\beta}_2 + 408\tilde{\beta}_2\tilde{\beta}_1 + 456\tilde{\beta}_1 + 213\},$$

$$\tau_3' - \tau_3 + 3p\tau_2 - 6(\tau_2' - \tau_2) - p^2(p+18) = \tfrac{1}{8}\{-24\tilde{\beta}_6 + 12\tilde{\beta}_4(13\tilde{\beta}_2 - 9) + 576\tilde{\beta}_3 - 166\tilde{\beta}_2^3 + 174\tilde{\beta}_2^2 - 936\tilde{\beta}_2\tilde{\beta}_1 + 186\tilde{\beta}_2 - 1080\tilde{\beta}_1 - 474\},$$

$$\tau_3 - 2\tau_3' - 2p(\tau_2 - \tau_2') - 8(\tau_2 - 2\tau_2') + 3p^3 - 32 = \tfrac{1}{8}\{8\tilde{\beta}_6 - 8\tilde{\beta}_4(9\tilde{\beta}_2 - 5) - 192\tilde{\beta}_3 + 432\tilde{\beta}_2\tilde{\beta}_1 + 90\tilde{\beta}_2^3 - 126\tilde{\beta}_2^2 + 198\tilde{\beta}_2 + 336\tilde{\beta}_1 - 138\}$$.....(39).

Therefore, from the equations (37), (38) and (39), we have

$$\text{Mean } \sigma = \tilde{\sigma}\left\{1 - \frac{1}{8N}(\tilde{\beta}_2 + 3) + \frac{1}{128N^2}(8\tilde{\beta}_4 - 15\tilde{\beta}_2^2 + 14\tilde{\beta}_2 - 48\tilde{\beta}_1 - 55) - \frac{1}{1024N^3}[40\tilde{\beta}_6 - 8\tilde{\beta}_4(35\tilde{\beta}_2 - 19) - 960\tilde{\beta}_3 + 315\tilde{\beta}_2^3 + 1680\tilde{\beta}_2\tilde{\beta}_1 - 285\tilde{\beta}_2^2 - 255\tilde{\beta}_2 + 1968\tilde{\beta}_1 + 1017]\right\},$$

$$\mu_2(\sigma) = \frac{\tilde{\sigma}^2}{4N}\left\{\tilde{\beta}_2 - 1 - \frac{1}{8N}(4\tilde{\beta}_4 - 7\tilde{\beta}_2^2 + 10\tilde{\beta}_2 - 24\tilde{\beta}_1 - 23) + \frac{1}{32N^2}[10\tilde{\beta}_6 - 4\tilde{\beta}_4(17\tilde{\beta}_2 - 11) - 240\tilde{\beta}_3 + 75\tilde{\beta}_2^3 - 79\tilde{\beta}_2^2 - 67\tilde{\beta}_2 + 408\tilde{\beta}_2\tilde{\beta}_1 + 456\tilde{\beta}_1 + 213]\right\},$$

$$\mu_3(\sigma) = \frac{\tilde{\sigma}^3}{16N^2}\left\{2\tilde{\beta}_4 - 3\tilde{\beta}_2^2 - 12\tilde{\beta}_1 + 1 - \frac{1}{16N}[24\tilde{\beta}_6 - 12\tilde{\beta}_4(13\tilde{\beta}_2 - 9) - 576\tilde{\beta}_3 + 166\tilde{\beta}_2^3 - 174\tilde{\beta}_2^2 + 936\tilde{\beta}_2\tilde{\beta}_1 - 186\tilde{\beta}_2 + 1080\tilde{\beta}_1 + 474]\right\},$$

and $$\mu_4(\sigma) = \frac{3\tilde{\sigma}^4}{16N^2}\left\{(\tilde{\beta}_2 - 1)^2 + \frac{1}{8N}[8\tilde{\beta}_6 - 8\tilde{\beta}_4(9\tilde{\beta}_2 - 5) - 192\tilde{\beta}_3 + 432\tilde{\beta}_2\tilde{\beta}_1 + 90\tilde{\beta}_2^3 - 126\tilde{\beta}_2^2 + 198\tilde{\beta}_2 + 336\tilde{\beta}_1 - 138]\right\}$$ .....(40).

These special formulae (40) correspond, when expressed in terms of semi-invariants, with Dr Craig's formulae for $\lambda_r(\sigma)$ $(r = 1, 2, 3, 4)$*.

* See Craig, *Metron*, Vol. VII. No. 4, p. 56, also *Biometrika*, Vol. XXI. pp. 287—293.

(11) Now let us consider the case where the parent distribution is normal.

In this case $\tilde{\beta}_1=\tilde{\beta}_3=\tilde{\beta}_5=\ldots=0,$

$\tilde{\beta}_2=3, \quad \tilde{\beta}_4=15, \quad \tilde{\beta}_6=105,$

and $\tilde{\beta}_8=945.$

Consequently we have, from the equations (25) and (30),

$$m_2=\frac{1}{2N}\left(1+\frac{1}{N}+\frac{1}{N^2}+\frac{1}{N^3}\right), \quad m_3=\frac{1}{N^2}\left(1+\frac{2}{N}+\frac{3}{N^2}\right),$$

$$m_4=\frac{3}{4N^2}\left(1+\frac{6}{N}+\frac{15}{N^2}\right), \qquad m_5=\frac{5}{8N^3},$$

$$m_6=\frac{5}{8N^3}\left(3+\frac{61}{N}\right), \qquad m_7=\frac{105}{4N^4},$$

$$m_8=\frac{105}{16N^4} \qquad \ldots\ldots(41),$$

and also $$a=1-\frac{1}{4N}-\frac{7}{32N^2}-\frac{19}{128N^3}-\frac{101}{2048N^4},$$

$$a'=\frac{1}{N}\left(1+\frac{3}{4N}+\frac{17}{32N^2}+\frac{49}{128N^3}\right) \qquad \ldots\ldots(42).$$

Consequently, from the equations (14′), we have

$$\text{Mean } \sigma=\tilde{\sigma}\left(1-\frac{3}{4N}-\frac{7}{32N^2}-\frac{9}{128N^3}-\frac{59}{2048N^4}\right) \qquad \ldots\ldots(43\,a),$$

$$\mu_2(\sigma)=\frac{\tilde{\sigma}^2}{2N}\left(1-\frac{1}{4N}-\frac{3}{8N^2}-\frac{27}{64N^3}\right) \qquad \ldots\ldots(43\,b),$$

$$\mu_3(\sigma)=\frac{\tilde{\sigma}^3}{4N^2}\left(1+\frac{3}{4N}+\frac{11}{32N^2}\right) \qquad \ldots\ldots(43\,c),$$

$$\mu_4(\sigma)=\frac{3\tilde{\sigma}^4}{4N^2}\left(1-\frac{1}{2N}-\frac{7}{16N^2}\right) \qquad \ldots\ldots(43\,d).$$

If we find the values of p, τ's and τ''s in this case we have:

$$p=1, \quad \tau_2=7, \quad \tau_3=19, \quad \tau_4=101, \quad \tau_2'=6, \quad \tau_3'=17 \text{ and } \tau_4'=98.$$

Substituting these values into the general approximate formulae (35), we can also deduce the same equations as (43) directly.

Now, from the equations (43), if we find the expressions for σ_σ, $\beta_1(\sigma)$ and $\beta_2(\sigma)$, we get

$$\sigma_\sigma=\frac{\tilde{\sigma}}{\sqrt{2N}}\left(1-\frac{1}{4N}-\frac{3}{8N^2}-\frac{27}{64N^3}\right)^{\frac{1}{2}} \qquad \ldots\ldots(44\,a)$$

$$=\frac{\tilde{\sigma}}{\sqrt{2N}}\left(1-\frac{1}{8N}-\frac{25}{128N^2}-\frac{241}{1024N^3}\right),$$

$$\beta_1(\sigma)=\frac{1}{2N}\left(1+\frac{9}{4N}+\frac{31}{8N^2}\right) \qquad \ldots\ldots(44\,b),$$

and $$\beta_2(\sigma)=3+\frac{3}{4N^2} \qquad \ldots\ldots(44\,c).$$

(12) *Special cases (continued).*

Finally let us consider the case where the parent distribution is of the form of Pearson's Type III curve, i.e.

$$y = y_0 \left(1 + \frac{x}{a}\right)^{\gamma a} e^{-\gamma x} \quad \text{.....(45),}$$

which is a skew distribution, and also the case of the exponential function (known as Pearson's Type X curve),

$$y = y_0 e^{-\frac{x}{\bar{x}}} \quad \text{.....(46),}$$

which is a special case of the distribution (45).

In these cases $\tilde{\beta}_2 = \frac{3}{2}(\tilde{\beta}_1 + 2)$, $\tilde{\beta}_1 = \dfrac{4}{\gamma a + 1}$,

and generally $\tilde{\beta}_{2m} = \dfrac{2m+1}{2}\{\tilde{\beta}_{2m-1} + 2\tilde{\beta}_{2m-2}\}$,

$$\tilde{\beta}_{2m+1} = (m+1)\{\tilde{\beta}_1\tilde{\beta}_{2m} + 2\tilde{\beta}_{2m-1}\} \quad \text{.....(47).}$$

From the equations (47) we get

$$\tilde{\beta}_3 = 3\tilde{\beta}_1^2 + 10\tilde{\beta}_1, \quad \tilde{\beta}_4 = \tfrac{5}{2}(3\tilde{\beta}_1^2 + 13\tilde{\beta}_1 + 6),$$

$$\tilde{\beta}_5 = \tfrac{3}{2}(15\tilde{\beta}_1^3 + 77\tilde{\beta}_1^2 + 70\tilde{\beta}_1),$$

$$\tilde{\beta}_6 = \tfrac{7}{4}(45\tilde{\beta}_1^3 + 261\tilde{\beta}_1^2 + 340\tilde{\beta}_1 + 60),$$

$$\tilde{\beta}_7 = 315\tilde{\beta}_1^4 + 2007\tilde{\beta}_1^3 + 3304\tilde{\beta}_1^2 + 1260\tilde{\beta}_1,$$

$$\tilde{\beta}_8 = \tfrac{9}{4}(630\tilde{\beta}_1^4 + 4329\tilde{\beta}_1^3 + 8435\tilde{\beta}_1^2 + 4900\tilde{\beta}_1 + 420) \quad \text{.....(48).}$$

On substituting these values for $\tilde{\beta}_m$ into the expressions for p, and the τ's and τ''s, we get, after simplification,

$$p = \tfrac{1}{4}(3\tilde{\beta}_1 + 4), \quad \tau_2 = -\tfrac{1}{16}(105\tilde{\beta}_1^2 + 344\tilde{\beta}_1 - 112),$$

$$\tau_2' = -\tfrac{1}{8}(33\tilde{\beta}_1^2 + 104\tilde{\beta}_1 - 48),$$

$$\tau_3 = \tfrac{1}{64}(8505\tilde{\beta}_1^3 + 3{,}786\tilde{\beta}_1^2 + 1{,}6304\tilde{\beta}_1 + 1216),$$

$$\tau_3' = \tfrac{1}{64}(3915\tilde{\beta}_1^3 + 1{,}7484\tilde{\beta}_1^2 + 7952\tilde{\beta}_1 + 1088),$$

$$\tau_4 = -\tfrac{1}{256}(654{,}2235\tilde{\beta}_1^4 + 3489{,}9984\tilde{\beta}_1^3 + 3571{,}9200\tilde{\beta}_1^2 + 313{,}7792\tilde{\beta}_1 - 2{,}5856),$$

and $\tau_4' = -\tfrac{1}{256}(126{,}5670\tilde{\beta}_1^4 + 680{,}7360\tilde{\beta}_1^3 + 716{,}1536\tilde{\beta}_1^2 + 71{,}4752\tilde{\beta}_1 - 2{,}5088)$

.....(49).

Now, from these equations and the equations (35), we can deduce easily

$$\text{Mean}\,\sigma=\tilde{\sigma}\left\{1-\frac{3\tilde{\beta}_1+12}{16N}+\frac{105\tilde{\beta}_1^2+392\tilde{\beta}_1-112}{512N^2}\right.$$
$$\left.-\frac{8505\tilde{\beta}_1^3+3,8700\tilde{\beta}_1^2+1,8864\tilde{\beta}_1+576}{8192N^3}\right.$$
$$\left.+\frac{654,2235\tilde{\beta}_1^4+3517,2144\tilde{\beta}_1^3+3691,7280\tilde{\beta}_1^2+362,1632\tilde{\beta}_1+1,5104}{52,4288N^4}\right\},$$

$$\mu_2(\sigma)=\frac{\tilde{\sigma}^2}{8N}\left\{3\tilde{\beta}_1+4-\frac{57\tilde{\beta}_1^2+232\tilde{\beta}_1+16}{16N}+\frac{2205\tilde{\beta}_1^3+1,0284\tilde{\beta}_1^2+5808\tilde{\beta}_1-192}{128N^2}\right.$$
$$\left.-\frac{83,1915\tilde{\beta}_1^4+451,5888\tilde{\beta}_1^3+489,1424\tilde{\beta}_1^2+55,5776\tilde{\beta}_1+6912}{4096N^3}\right\},$$

$$\mu_3(\sigma)=\frac{\tilde{\sigma}^3}{64N^2}\left\{33\tilde{\beta}_1^2+104\tilde{\beta}_1+16-\frac{2781\tilde{\beta}_1^3+1,3212\tilde{\beta}_1^2+8304\tilde{\beta}_1-192}{16N}\right.$$
$$\left.+\frac{108,8955\tilde{\beta}_1^4+596,0208\tilde{\beta}_1^3+663,7088\tilde{\beta}_1^2+87,8848\tilde{\beta}_1+2816}{512N^2}\right\},$$

$$\mu_4(\sigma)=\frac{\tilde{\sigma}^4}{64N^2}\left\{27\tilde{\beta}_1^2+72\tilde{\beta}_1+48+\frac{495\tilde{\beta}_1^3+1740\tilde{\beta}_1^2-240\tilde{\beta}_1-192}{8N}\right.$$
$$\left.-\frac{162,3348\tilde{\beta}_1^4+864,6336\tilde{\beta}_1^3+878,0160\tilde{\beta}_1^2+67,8912\tilde{\beta}_1+2,1504}{1024N^2}\right\}$$
.....(50).

These results correspond to those given by Craig*, except that in each formula I have proceeded to one degree higher approximation.

Further, if the parent distribution (45) reduce to the particular form

$$y=y_0e^{-\frac{x}{\tilde{\sigma}}},$$

then the $\tilde{\beta}$'s become

$$\tilde{\beta}_1=4,\quad \tilde{\beta}_2=9,\quad \tilde{\beta}_3=88,\quad \tilde{\beta}_4=265,\quad \tilde{\beta}_5=3708,\quad \tilde{\beta}_6=1,4833,$$
$$\tilde{\beta}_7=26,6992,\quad \tilde{\beta}_8=133,4961;$$

consequently

$$p=4,\quad \tau_2=-184,\quad \tau_2'=-112,\quad \tau_3=1,9008,\quad \tau_3'=8800,$$
$$\tau_4=-1754,8608,\quad \tau_4'=-342,6176,$$

and, from the equations (50), we have

$$\text{Mean}\,\sigma=\left(1-\frac{3}{2N}+\frac{49}{8N^2}-\frac{2421}{16N^3}+\frac{110,6203}{128N^4}\right)\tilde{\sigma},$$
$$\mu_2(\sigma)=\frac{1}{N}\left(2-\frac{29}{2N}+\frac{321}{N^2}-\frac{14,2207}{8N^3}\right)\tilde{\sigma}^2,$$
$$\mu_3(\sigma)=\frac{1}{N^2}\left(15-\frac{825}{2N}+\frac{18,7973}{8N^2}\right)\tilde{\sigma}^3,$$
$$\mu_4(\sigma)=\frac{1}{N^2}\left(12+\frac{114}{N}-\frac{6,7881}{4N^2}\right)\tilde{\sigma}^4$$
.....(51);

which give us the Mean σ, $\mu_r(\sigma)$ $(r=2, 3, 4)$ in sampling when the parent distribution is of the form of Pearson's Type X curve.

* *Biometrika*, Vol. xxi. p. 292, equation (17).

SECTION III. DEGREE OF APPROXIMATION. NUMERICAL VERIFICATION.

(13) If the parent distribution be normal and infinite, it is well known that the sampling distribution of σ is theoretically given by

$$y = A\sigma^{n-1} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (52),$$

where n is the size of repeated random samples, i.e.

$$n = N \text{ in foregoing articles,}$$

and if M be the total number of samples

$$A = \frac{M}{(n-3)(n-5)\ldots 3.1}\sqrt{\frac{2}{\pi}}\left(\frac{\tilde{\sigma}}{\sqrt{n}}\right)^{-(n-1)} \text{ if } n \text{ is even,}$$

or

$$\frac{M}{(n-3)(n-5)\ldots 4.2}\left(\frac{\tilde{\sigma}}{\sqrt{n}}\right)^{-(n-1)} \text{ if } n \text{ is odd.}$$

We may now compare the moments calculated from the equations (43) with the true moments obtained from the distribution law (52).

If μ_m' is the mth moment coefficient of (52) about $\sigma = 0$, then

$$\begin{aligned} M\mu_m' &= A\int_0^\infty \sigma^{m+n-2} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma \\ &= -A\left(\frac{\tilde{\sigma}^2}{n}\right)\left\{\left[\sigma^{n+m-3} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}}\right]_0^\infty - (n+m-3)\int_0^\infty \sigma^{m+n-4} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma\right\} \\ &= A(m+n-3)\left(\frac{\tilde{\sigma}^2}{n}\right)\int_0^\infty \sigma^{m+n-4} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma \\ &= A(n+m-3)(n+m-5)\left(\frac{\tilde{\sigma}^2}{n}\right)^2\int_0^\infty \sigma^{m+n-6} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma. \end{aligned}$$

Similarly, after integrating by parts k times, we have

$$\mu_m' = \frac{(n+m-3)(n+m-5)\ldots(n+m-2k-1)}{(n-3)(n-5)\ldots 3.1 \begin{cases}\sqrt{\pi/2}\\ \text{or } 4.2\end{cases}}\left(\frac{\tilde{\sigma}}{\sqrt{n}}\right)^{2k-n+1} \times \int_0^\infty \sigma^{n+m-2(k+1)} e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}} d\sigma \quad \ldots\ldots\ldots (53),$$

according as n is even or odd.

From this general formula we can easily deduce

$$\mu_1' = \frac{(n-2)(n-4)\ldots 3.1}{(n-3)(n-5)\ldots 4.2}\sqrt{\frac{\pi}{2n}}\,\tilde{\sigma} \text{ if } n \text{ is odd,}$$

or

$$\frac{(n-2)(n-4)\ldots 4.2}{(n-3)(n-5)\ldots 3.1\sqrt{\pi/2}}\,\tilde{\sigma} \text{ if } n \text{ is even} \quad \ldots\ldots\ldots\ldots\ldots (54\,a),$$

$$\mu_2' = \left(1-\frac{1}{n}\right)\tilde{\sigma}^2, \quad \mu_4' = \left(1-\frac{1}{n^2}\right)\tilde{\sigma}^4 \quad \ldots\ldots\ldots\ldots\ldots (54\,b),$$

and

$$\mu_3' = \tilde{\sigma}^2\mu_1' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (55),$$

which correspond to the special case of my formulae (12) when the parent distribution is normal, i.e.

$$\mu_1' = a_1\sqrt{\bar{\mu}_2} = \left(\frac{n-1}{n}\right)^{\frac{1}{2}} a\tilde{\sigma} \quad \text{.....(56 } a\text{)},$$

$$\mu_2' = a_2\bar{\mu}_2 = \left(1-\frac{1}{n}\right)\tilde{\sigma}^2 \quad \text{.....(56 } b\text{)},$$

$$\mu_3' = a_3(\bar{\mu}_2)^{\frac{3}{2}} = \left(\frac{n-1}{n}\right)^{\frac{3}{2}}(a+a')\tilde{\sigma}^3 \quad \text{.....(56 } c\text{)},$$

$$\mu_4' = a_4(\bar{\mu}_2)^2 = \left(\frac{n-1}{n}\right)^2(1+4m_2)\tilde{\sigma}^4 \quad \text{.....(56 } d\text{)},$$

where $m_2 = \dfrac{1}{2(n-1)}$,

and $$a = 1-\frac{1}{4n}-\frac{7}{32n^2}-\frac{19}{128n^3}-\frac{101}{2048n^4}-\cdots,$$

$$a' = \frac{1}{n}\left(1+\frac{3}{4n}+\frac{17}{32n^2}+\frac{49}{128n^3}+\cdots\right),$$

as we found already in Art. (11).

Now, since the equation (56 d) can be transformed into

$$\mu_4' = \left(1-\frac{1}{n^2}\right)\tilde{\sigma}^4,$$

we can see that my formulae for μ_2' and μ_4', when the parent distribution is normal, are identically equal to (54 b) respectively, and we have only to examine the accuracy of μ_1' and μ_3' in (56).

Now if we insert the expressions for a and a' into equations (56 a) and (56 c), retaining terms up to the order n^{-4}, we have

$$\mu_1' = \left(1-\frac{3}{4n}-\frac{7}{32n^2}-\frac{9}{128n^3}+\frac{59}{2048n^4}\right)\tilde{\sigma},$$

and
$$\mu_3' = \left(1-\frac{1}{n}\right)^{\frac{3}{2}}\left(1+\frac{3}{4n}+\frac{17}{32n^2}+\frac{49}{128n^3}+\frac{683}{2048n^4}\right)\tilde{\sigma}^3$$

$$= \left(1-\frac{3}{4n}-\frac{7}{32n^2}-\frac{9}{128n^3}+\frac{59}{2048n^4}\right)\tilde{\sigma}^3 \quad \text{.....(57)},$$

therefore $\mu_3' = \tilde{\sigma}^2\mu_1'$.

Accordingly, as far as terms of order n^{-4}, my formulae also lead to the relation (55), holding between the first and third moment coefficients of the distribution law (52) about $\sigma = 0$. Thus the accuracy of my formulae, for the special case of normal parent distribution, depends mainly upon the accuracy of the formula for μ_1'.

Now, from the equation (54 *a*), we have

$$\mu_1' = \frac{2^{n-2}\left[\left(\frac{n}{2}-1\right)!\right]^2}{(n-2)!}\sqrt{\frac{2}{n\pi}}\,\tilde{\sigma} \text{ if } n \text{ is even,}$$

or

$$\frac{(n-2)(n-3)!}{2^{n-3}\left[\left(\frac{n-3}{2}\right)!\right]^2}\sqrt{\frac{\pi}{2n}}\,\tilde{\sigma} \text{ if } n \text{ is odd} \quad \ldots\ldots\ldots\ldots\ldots\ldots(58).$$

If we apply to these expressions Stirling's formula

$$n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{5{,}1840n^3} - \frac{571}{248{,}8320n^4} + \ldots\right),$$

after calculation and simplification we obtain

$$\mu_1' = \tilde{\sigma}\left(1 - \frac{3}{4n} - \frac{7}{32n^2} - \frac{9}{128n^3} + \frac{59}{2048n^4} + \ldots\right),$$

which shows us how far my formula for μ_1' is accurate.

(14) Finally let us examine the accuracy of the formulae (43) for the moments of σ, and (44) for σ_σ, $\beta_1(\sigma)$ and $\beta_2(\sigma)$, numerically for various values of n, the size of sample.

Now if we write the equation (54 *a*) as follows:

$$\text{Mean } \sigma = \mu_1' = u\tilde{\sigma},$$

then

$$\mu_2(\sigma) = \left(1 - \frac{1}{n} - u^2\right)\tilde{\sigma}^2,$$

$$\sigma_\sigma = \sqrt{\left(1 - \frac{1}{n} - u^2\right)}\,.\,\tilde{\sigma},$$

which correspond to my formulae (43 *a*), (43 *b*) and (44 *a*).

The numbers in Tables I, II and III are the values of $\bar{\sigma}$, $\mu_2(\sigma)$ and σ_σ respectively, obtained from the equations above and my corresponding formulae.

For instance, in Table I the "accurate values" are those of the Mean σ, given by equation (54 *a*), and the values in the third column are for $\bar{\sigma}$ given by the formula (43 *a*).

The values in the fourth and fifth columns are for $\bar{\sigma}$, given also by the formula (43 *a*) when terms of order n^{-4} and terms of order n^{-3} respectively are neglected, i.e. by

$$\bar{\sigma} = \left(1 - \frac{3}{4n} - \frac{7}{32n^2} - \frac{9}{128n^3}\right)\tilde{\sigma},$$

and

$$\bar{\sigma} = \left(1 - \frac{3}{4n} - \frac{7}{32n^2}\right)\tilde{\sigma}.$$

The second of these equations is a special case of the formula (21 *a*) in Art. (5) when the parent distribution is normal, and the first is the special case which corresponds to Dr Craig's formula for $\lambda_1(\sigma)$ in terms of semi-invariants.

Comparing the values, given by my formulae, with the corresponding accurate values in these tables, we can see that my formulae up to the order n^{-4} are very accurate, even when $n < 10$.

TABLE I.

Values of $\frac{\overline{\sigma}}{\tilde{\sigma}}$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^4$	$1/n^3$	$1/n^2$
5	·840 7487	·840 7336	·840 6875	·841 2500
7	·888 2029	·888 1999	·888 1879	·888 3929
9	·913 8749	·913 8740	·913 8696	·913 9661
10	·922 7456	·922 7451	·922 7422	·922 8125
15	·949 0076	·949 0075	·949 0069	·949 0278
20	·961 9445	·961 9445	·961 9443	·961 9531
25	·969 6456	·969 6456	·969 6455	·969 6500
30	·974 7544	·974 7544	·974 7543	·974 7569
50	·984 9119	·984 9119	·984 9119	·984 9125
75	·989 9609	·989 9609	·989 9609	·989 9611
100	·992 4781	·992 4781	·992 4781	·992 4781
150	·994 9903	·994 9903	·994 9903	·994 9903
200	·996 2445	·996 2445	·996 2445	·996 2445

TABLE II.

Values of $\frac{\mu_2(\sigma)}{\tilde{\sigma}^2}$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^4$	$1/n^3$	$1/n^2$
5	·093 1417	·093 1625	·093 5000	·095 0000
7	·068 2385	·068 2431	·068 3309	·068 8776
9	·053 7216	·053 7230	·053 7551	·054 0123
10	·048 5405	·048 5414	·048 5625	·048 7500
15	·032 7179	·032 7181	·032 7222	·032 7778
20	·024 6627	·024 6627	·024 6641	·024 6875
25	·019 7875	·019 7875	·019 7880	·019 8000
30	·016 5206	·016 5206	·016 5208	·016 5278
50	·009 9485	·009 9485	·009 9485	·009 9500
75	·006 6440	·006 6440	·006 6440	·006 6444
100	·004 9873	·004 9873	·004 9873	·004 9875
150	·003 3277	·003 3277	·003 3277	·003 3278
200	·002 4969	·002 4969	·002 4969	·002 4969

TABLE III.

Values of $\sigma_\sigma/\tilde{\sigma}$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^{\frac{7}{2}}$	$1/n^{\frac{5}{2}}$	$1/n^{\frac{3}{2}}$
5	·3052	·3053	·3059	·3083
7	·2612	·2612	·2614	·2625
9	·2318	·2318	·2319	·2324
10	·2203	·2203	·2204	·2208
15	·1809	·1809	·1809	·1811
20	·1570	·1570	·1570	·1571
25	·1407	·1407	·1407	·1407
30	·1285	·1285	·1285	·1286
50	·0997	·0997	·0997	·0998
75	·0815	·0815	·0815	·0815
100	·0706	·0706	·0706	·0706
150	·0577	·0577	·0577	·0577
200	·0500	·0500	·0500	·0500

For the calculation of $\bar{\sigma}$, $\mu_2(\sigma)$ and σ_σ, the simple formulae

$$\left\{\begin{aligned} \bar{\sigma} &= \left(1 - \frac{3}{4n} - \frac{7}{32n^2}\right)\tilde{\sigma}, \\ \mu_2(\sigma) &= \frac{1}{2n}\left(1 - \frac{1}{4n}\right)\tilde{\sigma}^2, \\ \sigma_\sigma &= \frac{1}{\sqrt{2n}}\left(1 - \frac{1}{8n}\right)\tilde{\sigma} \end{aligned}\right. \quad \text{.....(22 bis),}$$

and also

$$\left\{\begin{aligned} \bar{\sigma} &= \left(1 - \frac{3}{4n} - \frac{7}{32n^2} - \frac{9}{128n^3}\right)\tilde{\sigma}, \\ \mu_2(\sigma) &= \frac{1}{2n}\left(1 - \frac{1}{4n} - \frac{3}{8n^2}\right)\tilde{\sigma}^2, \\ \sigma_\sigma &= \frac{1}{\sqrt{2n}}\left(1 - \frac{1}{8n} - \frac{25}{128n^2}\right)\tilde{\sigma} \end{aligned}\right. \quad \text{.....(59),}$$

give us good estimates when n is not very small, as we can see in the tables, and we may therefore use these simple formulae in the calculation of $\bar{\sigma}$, $\mu_2(\sigma)$ and σ_σ, provided n is not small.

Next, let us examine the values of $\mu_3(\sigma)$ and $\mu_4(\sigma)$. From the equations (54), (55) and well-known relations between μ_r' and μ_r, moment coefficients about the mean, we can deduce

$$\mu_3(\sigma) = \frac{\bar{\sigma}}{n}\left\{1 - \frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right\}\tilde{\sigma}^2,$$

and $$\mu_4(\sigma) = \left\{\frac{5}{4n^2} - \frac{1}{2n}\left(4 - \frac{3}{n}\right)\left(1 - \frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right) - \frac{3}{4n^2}\left(1 - \frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right)^2\right\}\tilde{\sigma}^4 \text{...(60)*},$$

* These non-approximate equations were first deduced by K. Pearson; see *Biometrika*, Vol. x. p. 526. They were used to table the accurate values of $\beta_1(\sigma)$, $\beta_2(\sigma)$ for various sized samples.

which correspond to my formulae (43) for $\mu_3(\sigma)$ and $\mu_4(\sigma)$, i.e.

$$\mu_3(\sigma) = \frac{\tilde{\sigma}^3}{4n^2}\left(1 + \frac{3}{4n} + \frac{11}{32n^2}\right),$$

$$\mu_4(\sigma) = \frac{3\tilde{\sigma}^4}{4n^2}\left(1 - \frac{1}{2n} - \frac{7}{16n^2}\right) \quad \text{.....(43 } bis\text{)},$$

as already found in Art. (11).

I have calculated the values of $\mu_3(\sigma)$ and $\mu_4(\sigma)$ given by (60), and also those given by (43 *bis*), and they are shown in the second and the third columns of Tables IV and V.

We can obtain also two degrees of approximation more, as before, according to whether we neglect terms of the equations (43 *bis*), of order $\frac{1}{n^4}$, or also those of order $\frac{1}{n^3}$. These values are given in the fourth and five columns of Tables IV and V.

It will be seen that the complete formulae (43 *bis*) give us very good approximations, even for quite small samples, while the approximation which neglects the terms in n^{-4} will often be quite adequate.

But the simplest formulae

$$\mu_3(\sigma) = \frac{1}{4n^2}\tilde{\sigma}^3$$

and

$$\mu_4(\sigma) = \frac{3}{4n^2}\tilde{\sigma}^4$$

do not give good results unless n is very large, and when it is the distribution of σ has become practically a normal curve.

TABLE IV.

Values of $\mu_3(\sigma)/\tilde{\sigma}^3$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^4$	$1/n^3$	$1/n^2$
5	·011 5323	·011 6375	·011 5000	·010 0000
7	·005 6669	·005 6845	·005 6487	·005 1020
9	·003 3521	·003 3567	·003 3436	·003 0864
10	·002 6934	·002 6961	·002 6875	·002 5000
15	·001 1680	·001 1684	·001 1667	·001 1111
20	·000 6489	·000 6490	·000 6484	·000 6250
25	·000 4122	·000 4122	·000 4120	·000 4000
30	·000 2848	·000 2848	·000 2847	·000 2778
50	·000 1015	·000 1015	·000 1015	·000 1000
75	·000 0449	·000 0449	·000 0449	·000 0444
100	·000 0252	·000 0252	·000 0252	·000 0250
150	·000 0112	·000 0112	·000 0112	·000 0111
200	·000 0063	·000 0063	·000 0063	·000 0063

TABLE V.

Values of $\mu_4(\sigma)/\tilde{\sigma}^4$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^4$	$1/n^3$	$1/n^2$
5	·026 541	·026 475	·027 000	·030 000
7	·014 086	·014 076	·014 213	·015 306
9	·008 697	·008 695	·008 745	·009 259
10	·007 094	·007 092	·007 125	·007 500
15	·003 216	·003 216	·003 222	·003 333
20	·001 826	·001 826	·001 828	·001 875
25	·001 175	·001 175	·001 176	·001 200
30	·000 819	·000 819	·000 819	·000 833
50	·000 297	·000 297	·000 297	·000 306
75	·000 132	·000 132	·000 132	·000 133
100	·000 075	·000 075	·000 075	·000 075
150	·000 033	·000 033	·000 033	·000 033
200	·000 019	·000 019	·000 019	·000 019

Finally, let us examine the values of $\beta_1(\sigma)$ and $\beta_2(\sigma)$. From the equations (59) and (60) Prof. K. Pearson deduced

$$\beta_1(\sigma)=\frac{\tilde{\sigma}^2\tilde{\sigma}^4}{n^2\mu_2(\sigma)^3}\left(1-\frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right)^2,$$

and $$\beta_2(\sigma)=\left(\frac{\tilde{\sigma}^2}{2n\mu_2(\sigma)}\right)^2\left\{5-2n\left(4-\frac{3}{n}\right)\left(1-\frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right)-3\left(1-\frac{2n\mu_2(\sigma)}{\tilde{\sigma}^2}\right)^2\right\} \quad \ldots\ldots\ldots(61),$$

which give the values of the second columns of Tables VI and VII.

My corresponding formulae are

$$\beta_1(\sigma)=\frac{1}{2n}\left(1+\frac{9}{4n}+\frac{31}{8n^2}\right), \quad \beta_2(\sigma)=3+\frac{3}{4n^2} \quad \ldots\ldots\ldots(44\,bis),$$

as already found in Art. (11).

These expressions for β's lead to the figures in the third columns of Tables VI and VII, while the fourth and fifth columns result from the equations (44 *bis*), neglecting (i) the terms of the highest order in n^{-1}, and (ii) also the next highest order terms.

It will be seen, from these numerical values, that if n is not very small the accuracy of my full approximation for $\beta_1(\sigma)$ and $\beta_2(\sigma)$, as far as four decimal places, is very satisfactory. But the other less accurate approximations are not so and, especially, are quite inadequate for $\beta_2(\sigma)$ when n is small.

TABLE VI.

Values of $\beta_1(\sigma)$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order		
		$1/n^3$	$1/n^2$	$1/n$
5	·1646	·1605	·1450	·1000
7	·1011	·1000	·0944	·0714
9	·0725	·0721	·0694	·0556
10	·0634	·0632	·0613	·0500
15	·0390	·0389	·0383	·0333
20	·0281	·0281	·0278	·0250
25	·0219	·0219	·0218	·0200
30	·0180	·0180	·0179	·0167
50	·0105	·0105	·0105	·0100
75	·0069	·0069	·0069	·0067
100	·0051	·0051	·0051	·0050
150	·0038	·0038	·0038	·0033
200	·0028	·0028	·0028	·0025

TABLE VII.

Values of $\beta_2(\sigma)$.

n size of sample	Accurate values	Approximations, given by my formulae, including terms of order			From formula (62)
		$1/n^2$	$1/n$	$(1/n)^0$	
5	3·0593	3·0300	3·0000	3·0000	3·0488
7	3·0251	3·0153	3·0000	3·0000	3·0221
9	3·0136	3·0093	3·0000	3·0000	3·0125
10	3·0106	3·0075	3·0000	3·0000	3·0098
15	3·0042	3·0033	3·0000	3·0000	3·0040
20	3·0022	3·0019	3·0000	3·0000	3·0022
25	3·0014	3·0012	3·0000	3·0000	3·0014
30	3·0009	3·0008	3·0000	3·0000	3·0009
50	3·0003	3·0003	3·0000	3·0000	3·0003
75	3·0001	3·0001	3·0000	3·0000	3·0001
100	3·0001	3·0001	3·0000	3·0000	3·0001
150	3·0000	3·0000	3·0000	3·0000	3·0000
200	3·0000	3·0000	3·0000	3·0000	3·0000

[*Note.*] From our approximation we cannot find the term in n^{-3} for $\beta_2(\sigma)$ exactly, but if in expanding $(\mu_2(\sigma))^{-2}$ when using (43 *b*) to calculate β_2, we retain one more term in our calculation, we get

$$\beta_2(\sigma) = 3 + \frac{3}{4n^2}\left(1 + \frac{25}{8n}\right) \quad \text{.....(62).}$$

If we use this equation for the calculation of $\beta_2(\sigma)$, we get more accurate values of $\beta_2(\sigma)$, as indicated in the sixth column of Table VII, when the parent distribution is normal. But, from these results only, we cannot assert that this incomplete term in n^{-3} of $\beta_2(\sigma)$ should always be retained*.

(15) The numerical examination we have carried out in the foregoing article gives us some idea of the accuracy of the general approximate formulae (35), for if in the special case when the parent distribution is normal we get good results, we may anticipate a somewhat like adequacy in the more general result although the convergency is usually less satisfactory in non-normal cases.

Moreover, I may remark that calculations by the formulae (43) and (45) are all much simpler than those by the formulae deduced from the distribution law

$$y = A\sigma^{n-1}e^{-\frac{n\sigma^2}{2\tilde{\sigma}^2}}.$$

Accordingly formulae (43) and (45) are themselves of value not only as special cases of the general formulae (35), but also for their adaptability to simple calculations. Thus they may be useful even in the case when the sampled population is normal but the Table† of the constants of the σ distribution is not at hand.

(16) The general equation (15) was obtained subject to only one condition, namely that the expansion of $\left(1+\frac{y}{\bar{\mu}_2}\right)^{\frac{1}{2}}$ under the integral in a series of ascending powers of y was justifiable. This would be true if

$$\frac{|y|}{\bar{\mu}_2} < 1 \quad \text{or} \quad \frac{|\mu_2 - \bar{\mu}_2|}{\bar{\mu}_2} < 1 \quad \text{.....(63).}$$

Let us consider more carefully what this implies: Since $\mu_2 \geqq 0$ and $\bar{\mu}_2 = \tilde{\sigma}^2(N-1)/N$, the condition (63) may be written as follows:

$$\frac{N-1}{N}(1-\epsilon)\,\tilde{\sigma}^2 \leqslant \mu_2 \leqslant \frac{N-1}{N}(1+\epsilon)\,\tilde{\sigma}^2 \text{ .....(63')},$$

where ϵ is a positive quantity, less than 1, but $1-\epsilon$ may be as small as we please. This means that our expansion would be justified if we exclude values of μ_2 whose deviations from the mean, $\bar{\mu}_2 = \frac{N-1}{N}\tilde{\sigma}^2$, are numerically greater than

$$\epsilon\bar{\mu}_2 = \epsilon\frac{N-1}{N}\tilde{\sigma}^2.$$

Suppose l_1 and l_2 are the lower and upper limits of $y = \mu_2 - \bar{\mu}_2$ in sampling, l_1 will presumably be $-\bar{\mu}_2$ while l_2 may for certain theoretical distributions at any

* [To calculate the true term of $\beta_2(\sigma)$ in n^{-3} requires us to find μ_1' to the *fifth* order term, or another term, that in n^{-5}, in Stirling's Theorem. A better approximation than (62) is:

$$\beta_2(\sigma) = 3\left(1+\frac{1}{4n^2}+\frac{1}{n^3}\right).$$ Ed.]

† *Biometrika*, Vol. x. p. 528.

rate take the value ∞, but practically l_2 is finite; then in the equation (11), $a_i\,(i=1, 2, 3, 4)$ may be written

$$a_i=\int_{l_1}^{-\epsilon\bar{\mu}_2}\Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}dy+\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}dy+\int_{\epsilon\bar{\mu}_2}^{l_2}\Phi(y)\left(1+\frac{y}{\bar{\mu}_2}\right)^{i/2}dy$$

$$=f_1+f_2+f_3, \text{ say} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(64),$$

and, with the restriction (63), we retain only the second integral f_2 for a_i.

Thus we have first to consider the convergence of f_2.

Now
$$f_2=1+\sum_{r=1}^{\infty}\frac{C_{i.r}}{\bar{\mu}_2{}^r}\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(65)$$

and $\Phi(y)$ is always positive and in most cases limited.

Hence if M be the upper limit of Φ, then

$$\frac{1}{\bar{\mu}_2{}^r}\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy\leqslant\frac{1}{\bar{\mu}_2{}^r}\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}My^r\,dy\leqslant\frac{2M\bar{\mu}_2}{r+1}\epsilon^{r+1} \quad \ldots\ldots\ldots\ldots(66),$$

and therefore the first expression in (66) decreases as r increases. Further $C_{i.r}$ $(r\leqslant 4)$ is less than 1 for finite values of r, becomes alternatively positive and negative and is such that $\underset{r\to\infty}{\mathrm{Lim}}\ C_{i.(r+1)}/C_{i.r}=1$. Hence we can see that the series (65) for f_2 is convergent.

Secondly, if l_1, l_2 lie within the interval $(-\epsilon\bar{\mu}_2, \epsilon\bar{\mu}_2)$, then since $\Phi(y)=0$ always in the intervals $(-\epsilon\bar{\mu}_2, l_1)$ and $(\epsilon\bar{\mu}_2, l_2)$,

$$f_1=f_3=0,$$

and the equation (11) becomes quite exact. If l_1 is out of the interval $(-\epsilon\bar{\mu}_2, \epsilon\bar{\mu}_2)$, as $1-\epsilon$ may be as small as we please, l_1 must be $-\bar{\mu}_2$ and

$$f_1=\int_{-\bar{\mu}_2}^{-\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy\to 0 \text{ as } \epsilon\to 1.$$

Therefore we can say that f_1 is always negligible as compared with f_2.

Now ${}_2M_r$ being the rth moment coefficient about the mean of μ_2,

$${}_2M_r=\int_{l_1}^{l_2}\Phi(y)\,y^r\,dy=\int_{l_1}^{-\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy+\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy+\int_{\epsilon\bar{\mu}_2}^{l_2}\Phi(y)\,y^r\,dy$$

$$=s_r+\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy+s_r', \text{ say} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(67).$$

But we can easily show that s_r is negligible compared with the second integral as before.

Accordingly
$${}_2M_r=\int_{-\epsilon\bar{\mu}_2}^{\epsilon\bar{\mu}_2}\Phi(y)\,y^r\,dy+s_r' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(67').$$

It will be seen therefore that the justification of my equation (11), expressing $a_i\,(i=1, 2, 3, 4)$ in terms of m_r or ${}_2M_r$, is based upon the assumption that *the integrals f_3 in* (64) *and s_r' in* (67) *are negligible compared with f_2 and ${}_2M_r$ respectively.*

If $l_2 \leqslant \epsilon\bar{\mu}_2$, this assumption becomes needless. Generally speaking the question whether the condition be satisfied or not must depend for its answer upon the form of $\Phi(y)$, but to obtain the general form of Φ, for which s_r' and f_3 are negligible, is not an easy problem. We may note, however, that in usual cases for $|y| > \epsilon\bar{\mu}_2$, $\Phi(y)$ is very small, and since

$$\sigma_y = \frac{N-1}{N^{\frac{3}{2}}}\sqrt{\tilde{\beta}_2 - 3 + \frac{2N}{N-1}}\,\tilde{\sigma}^2 \to 0 \quad \text{as} \quad N \to \infty,$$

Φ will tend steadily to zero as the sample size is increased. The results, given in Art. (14), suggest that at any rate for populations in the neighbourhood of the normal, where we have a direct check, the formulae I have developed on this assumption provide good approximations to the moments of the standard deviation.

A STUDY OF SEVENTY-ONE NINTH DYNASTY EGYPTIAN SKULLS FROM SEDMENT.

By T. L. WOO, Ph.D.

1. *Source of the Material.* In the winter 1920—21 the British School of Archaeology in Egypt, directed by Sir Flinders Petrie, excavated a large cemetery at Gebel Sedment. This site is 70 miles south of Cairo and it overlooks the Fayum. A report on the excavations has been published*. Graves of various dynasties ranging from the 1st to the 19th were examined. The majority of the interments belong to the 9th dynasty and 71 well-preserved skulls of that date were kindly presented to the Biometric Laboratory. These form the subject of the present paper. As far as is known the crania preserved were not selected in any way except that preference was given to complete specimens. The grave furniture did not suggest that any foreign elements were present in the population represented.

2. *The Nature of the Series and Remarks on Individual Crania.* The series from Sedment consists of 71 skulls of which the majority are well preserved and almost complete. There are 62 more or less complete mandibles. Several specimens which were damaged in transportation have been repaired. In general appearance the skulls do not differ markedly from those belonging to other dynastic Egyptian series and they appear to belong to a single racial type. There is no reason to believe that the sample is more heterogeneous than normal ones representing a single cemetery and a restricted period of time. Only one skull (No. 28†) possesses Negroid features.

There are 69 fully adult specimens. One (No. 20) is juvenile and no use was made of its measurements. Another (No. 64) is not quite adult as the basal suture is open, although all the 3rd molars are fully erupted: its measurements were used in computing the ♂ means. The series was sexed anatomically by Dr Morant and he distinguished 40 males and 30 females. These sexes could be compared with ones given in the archaeological report in 28 cases: there is agreement in 25 and the evidence of the skull alone was accepted in the other 3 cases (Nos. 12, 45 and 67). Several ♂ specimens are exceptionally massive and muscular (especially Nos. 7 and 39) and one ♀ is unusually small and feeble.

* Sir Flinders Petrie and Guy Brunton: *Sedment. British School of Archaeology in Egypt and Egyptian Research Account.* 27th year, 1921. 2 Vols. London, 1924.

† The skull numbers given in the text are the serial ones which were marked on the specimens in the Biometric Laboratory. The corresponding grave numbers are given in the tables of individual measurements (Appendix I).

Remarks on individual crania are given in the tables of measurements (Appendix I). The condition of the coronal, sagittal and lambdoid sutures was examined and, unless otherwise stated, it may be assumed that these three at least are all open. In order to obtain a rough estimate of the age constitution of the sample, division was made into the three groups for which frequencies are given in the table below. The divisions between them are not precise but a clear sexual difference is indicated. It is known that sutural closing normally begins at an earlier age for male than for female skulls*, so this is no evidence of a sexual difference between the mean ages at death. Individuals at different adult stages of development are evidently represented.

Condition of coronal, sagittal and lambdoid sutures	Males	Females
All open	6 (15·0 %)	16 (53·4 %)
Beginning to close or partly closed	19 (47·5 %)	10 (33·3 %)
All closed or nearly closed ...	15 (37·5 %)	4 (13·3 %)

Thinning of the calvarial walls due to age was noted in the case of 1 ♂ skull (No. 71). In the present series the majority of the skulls show the sutures closing in a definite order, the sagittal being first, the coronal second and the lambdoid last. A few specimens show the coronal closing before the sagittal. The lambdoid suture frequently closes before the coronal in the case of Western European crania. Three specimens have a complete metopic suture, contact being made between the right frontal and left parietal bones in 2 cases (Nos. 22 ♂ and 48 ♂) and between the left frontal and right parietal in the other (No. 67 ♀). Faint traces of the frontal suture were also found on 2 other skulls (Nos. 19 ♂ and 52 ♂). The frequency of this anomaly (4·3 %) is almost exactly the same as for the Badari Egyptians† (5·1 %) but not a single instance of it was found among 47 skulls of the 1st Dynasty‡. Metopism is more frequently met with in European series, as several have provided percentages ranging between 7 and 10.

It is commonly found that one or more wormian bones are present in the lambdoid suture in considerably more than 50 % of the specimens forming an unselected European series. Judging from the small samples which have been examined, these supernumerary bones are less frequently met with among Egyptian crania. Of the 40 ♂ skulls from Sedment, 15 have wormians, and of the 30 ♀ skulls there are also 15 with one or more wormian bones. The following ossicles were found in other positions, the sexes being pooled: 7 of the lambda, 3 in

* See J. Frédéric: "Untersuchungen über die normale Obliteration der Schädelnähte." *Zeitschrift für Morphologie und Anthropologie*, Band IX. (1906), S. 373—456. See especially S. 439 and S. 442—443.

† B. Stoessiger: "A Study of the Badarian Crania recently excavated by the British School of Archaeology in Egypt." *Biometrika*, Vol. XIX. (1927), pp. 110—150.

‡ G. M. Morant: "A Study of Egyptian Craniology from Prehistoric to Roman Times." *Ibid.* Vol. XVII. (1925), pp. 1—52.

the sagittal, 2 in the coronal suture, 3 at the right asterion and 1 at both asteria. There are no ossicles of the bregma. Inter-parietal bones are also lacking and only 2 skulls (Nos. 59 and 65) show traces of the transverse occipital suture. Traces of the suture between the ex- and supra-occipital bones were also found in 2 cases (Nos. 30 and 49). There is one ♀ specimen with fronto-temporal articulation on the right side (No. 51) among 67 which could be examined for this feature. Epipteric bones appear to be unusually common: 9 cases of one or more epipteric bones on one or both sides were found among 32 ♂ specimens and 12 cases among 25 ♀ specimens. One ♂? skull (No. 63) has a large protruding ossicle above the right parieto-mastoid suture (see Plate V B).

In general the teeth are in a remarkably good state of preservation. A large proportion of the skulls, including ageing and aged specimens, was found with dentition complete and free from disease. Most of the teeth are markedly worn. A few cases were noted of adult skulls with no 3rd molars in one or both jaws. Three carious molars were the only diseased teeth found.

The frequency with which palate bridges occur was not observed, as experience has shown that they are often broken before the skull is examined in the laboratory and it is not possible then to tell whether the original bridge was complete or not. One unusual case of multiple bridges (No. 65) may be noted.

The existence of single or double precondyles on the basi-occipital was recorded. We found 5 cases of double and 3 of single precondyles.

There is one case of marked calvarial asymmetry (No. 26) and one of asymmetry of the facial skeleton (No. 58). The sizes of the jugular foramina were compared: *JR* denotes that the right is the greater, *JL* that the left is the greater and *J* = that no difference can be discerned between the sizes of the two. The following frequencies are given:

	Male	Female
JR	24	14
J=	3	7
JL	8	6

Every cranial series which has been examined in this way has shown that the right jugular foramen tends to be larger than the left. The question of calvarial asymmetry is discussed in the section on contour measurements below.

All cases of tympanic perforation have been recorded. There are 11 with perforation of the plate on both sides and 6 others with perforation on one side or the other. The percentage frequencies appear to be unusually high.

Few uncommon anomalies of particular interest were observed. One skull (No. 33) has a small exostosis on the supra-occipital, another (No. 10) has an excrescence behind the left condyle. One ♂ specimen (No. 65) has a markedly retreating frontal bone, a prominent superciliary ridge and also a basi-occipital incisure on the left side (see Plate V A).

3. *Comparison with other Series by the Method of the Coefficient of Racial Likeness.*

Measurements of the series were determined according to the technique used by previous workers in this Laboratory and the usual index letters are employed*. The ♂ and ♀ mean measurements are in Tables I and II respectively and the individual measurements in Appendix I. As the sample is a small one, all the standard deviations were not calculated, but comparison is made below between male values for six characters and the corresponding standard deviations which have been given for the series of 900 ♂ Egyptian skulls of the 26th—30th Dynasties†.

	L	B	H'	$100\,B/L$	$100\,H'/L$	$100\,B/H'$
9th Dynasty Egyptians from Sedment	5·85 ± ·44	3·98 ± ·30	4·36 ± ·39	2·71 ± ·20	2·99 ± ·23	3·55 ± ·27
26th—30th Dynasty Egyptians from Gizeh	5·72 ± ·09	4·76 ± ·08	5·03 ± ·08	2·68 ± ·06	2·94 ± ·05	4·30 ± ·06

This comparison suggests that the Sedment series is not more variable than the Gizeh one and the latter is known to be more homogeneous than most cranial series available.

Professor Karl Pearson's method of the coefficient of racial likeness was used to determine the racial affinities of our sample. This is defined for practical purposes to be:

$$\frac{1}{m}S\left[\frac{(M_s - M'_s)^2}{\sigma_s^2} \times \frac{n_s n'_s}{n_s + n'_s}\right] - 1 + \frac{1}{m} \pm \cdot 67449\sqrt{\frac{2}{m}},$$

where M_s is the mean based on n_s skulls for the first series, M'_s and n'_s are the corresponding constants for the second series and m characters are compared. The σ's of the long 26th—30th Dynasty Egyptian series were used throughout‡ and the coefficients were calculated for the 31 characters usually employed, or for as many of them as were available. The coefficient may be written:

$$\frac{1}{m}S(\alpha) - 1 + \frac{1}{m} \pm \cdot 67449\sqrt{\frac{2}{m}},$$

where

$$\alpha = \frac{(M_s - M'_s)^2}{\sigma_s^2} \times \frac{n_s n'_s}{n_s + n'_s}.$$

The mean number of skulls available for the characters used is denoted by $\bar{n}_s$ in the case of the first sample and $\bar{n}'_s$ in the case of the second sample and these "sizes" of the samples are usually unequal and may be of very different orders. The values of the coefficients are largely determined by the "sizes" of the

* Definitions of the measurements will be found in any recent volume of *Biometrika*.

† Karl Pearson and Adelaide G. Davin: "On the Biometric Constants of the Human Skull," *Biometrika*, Vol. XVI. (1924), pp. 328—363.

‡ *Ibid.* Vol. XVI. (1924), pp. 338 and 339.

TABLE I.

Mean Male Measurements of the Sedment and related Series.

Characters	9th Dynasty Egyptians: Sedment (Woo)	4th and 5th Dynasty Egyptians: Deshasheh and Medum (Thomson and MacIver)	Modern Cretans* (v. Luschan)	18th—20th Dynasty Egyptians: Thebes (Stahr)	Middle Dynastic Egyptians: El-Kubanieh North (Toldt)	Modern Egyptians*: Cairo (Schmidt)	Modern Abyssinians: Tigre district (Sergi)
C	1426·9 (35)	—	—	1451·0 (50)	1355·0 (27)	1348·0 (47)	—
F	179·6 (40)	183·0 (30)	—	—	—	—	—
L	181·9 (40)	184·9 (54)	180·2 (53)	183·5 (57)	182·2 (37)	176·1 (47)	183·7 (65)
B	138·3 (40)	139·3 (54)	139·4 (53)	137·3 (56)	134·2 (37)	137·6 (46)	136·3 (65)
B'	92·6 (40)	—	96·1 (52)	94·9 (50)	90·4 (37)	94·3 (47)	94·6 (65)
H	138·2 (38)	—	—	—	—	136·0 (46)	—
E'	137·4 (38)	136·0 (50)	137·5 (52)	133·7 (54)	135·9 (35)	134·6 (46)	133·5 (64)
OH	115·3 (39)	—	—	114·1 (56)	116·1 (36)	—	112·6 (65)
LB	100·8 (37)	101·8 (47)	101·5 (40)	101·9 (56)	101·6 (34)	100·9 (46)	99·8 (64)
Q'	314·9 (39)	—	—	313·5 (52)	—	312·9 (46)	309·8 (65)
S	373·8 (36)	—	—	372·1 (51)	372·7 (33)	364·8 (46)	372·7 (58)
S_1	129·0 (39)	—	—	125·7 (54)	126·0 (36)	—	130·2 (65)
S_2	129·9 (40)	—	—	129·8 (50)	130·9 (33)	—	128·9 (65)
S_2'	115·0 (37)	—	—	114·7 (54)	115·1 (30)	—	113·7 (57)
S_1'	113·3 (39)	—	—	111·9 (56)	—	—	112·9 (65)
S_2'	115·6 (40)	—	—	116·1 (52)	—	—	114·9 (65)
S_3'	98·6 (37)	—	—	96·1 (54)	97·3 (30)	95·4 (46)	95·8 (57)
U	511·7 (40)	—	—	512·6 (54)	505·6 (35)	501·3 (46)	513·0 (65)
$G'H$	71·5 (38)	71·3 (19)	69·1 (49)	70·1 (56)	70·1 (29)	69·8 (46)	69·1 (65)
GB	93·9 (37)	—	—	95·5 (54)	94·9 (29)	94·8 (43)	93·1 (57)
J	127·2 (29)	127·1 (16)	129·7 (51)	128·6 (53)	125·5 (26)	127·9 (44)	125·0 (42)
NH'	51·4 (38)	52·0 (44)	50·7 (49)	50·8 (56)	49·8 (37)	50·2 (46)	48·9 (64)
NB	24·5 (39)	25·1 (43)	24·2 (49)	25·3 (53)	24·8 (36)	24·8 (45)	25·0 (63)
DC	21·2 (32)	—	21·6 (48)	21·1 (52)	23·4 (34)	—	22·0 (65)
SC	9·5 (37)	—	—	10·8 (52)	—	—	9·5 (64)
O_1'	(R) 39·7 (35)	—	—	38·3 (59)	38·1 (35)‡	—	39·4 (65)
O_2	(R) 33·3 (40)	—	—	32·8 (53)	32·3 (35)	32·5 (46)	33·2 (65)
G_1'	46·7 (36)	—	—	—	46·4 (25)	46·6 (41)	—
G_2	40·1 (31)	—	—	40·5 (46)	—	40·3 (39)	39·0 (61)
GL	96·0 (37)	96·3 (42)	93·3 (32)	96·1 (53)	96·5 (25)	97·5 (45)	95·2 (64)
fml	35·9 (37)	—	—	35·2 (55)	34·9 (32)	35·0 (46)	36·1 (57)
fmb	30·1 (38)	—	—	29·7 (54)	29·2 (32)	29·0 (46)	29·5 (59)
$100\,B/L$	76·1 (40)	{75·4 (50)}†	77·5 (53)	74·8 (54)	73·7 (37)	{78·1 (46)}	74·3 (65)
$100\,H'/L$	75·5 (38)	{72·6 (50)}	{76·3 (52)}	72·8 (52)	74·7 (35)	{76·4 (46)}	72·7 (64)
$100\,B/H'$	100·9 (38)	{102·4 (50)}	{101·4 (52)}	{102·7 (54)}	{98·7 (35)}	{102·3 (46)}	{102·1 (64)}
$100(B-H')/L$	+0·6 (38)	{+1·8 (50)}	{+1·1 (52)}	{+2·0 (54)}	{−0·9 (35)}	{+1·7 (46)}	{+1·5 (64)}
$100\,G'H/GB$	76·3 (36)	—	—	73·9 (49)	{73·9 (29)}	73·7 (41)	73·9 (57)
$100\,NB/NH'$	47·7 (38)	{48·2 (44)}	47·7 (49)	50·5 (49)	49·9 (36)	49·4 (45)	51·1 (63)
$100\,O_2/O_1'$	(R) 84·1 (35)	—	—	86·5 (53)	84·9 (35)	—	84·3 (65)
$100\,G_2/G_1'$	85·6 (30)	—	—	—	—	87·0 (36)	—
$100\,fmb/fml$	84·1 (37)	—	—	{84·4 (54)}	84·0 (31)	83·0 (46)	{81·7 (57)}
$Oc.\,I.$	64·0 (37)	—	—	60·7 (54)	61·9 (30)	—	62·0 (57)
$N\angle$	65°·1 (36)	{64°·6 (19)}	{63°·3 (32)}	{64°·5 (53)}	{65°·2 (25)}	{66°·7 (45)}	{65°·4 (64)}
$A\angle$	72°·4 (36)	{73°·4 (19)}	{74°·6 (32)}	{74°·0 (53)}	{73°·4 (25)}	{72°·3 (45)}	{73°·2 (64)}
$B\angle$	42°·5 (36)	{42°·0 (19)}	{41°·9 (32)}	{41°·5 (53)}	{41°·4 (25)}	{41°·0 (45)}	{41°·4 (64)}
θ_1	30°·6 (36)	—	—	{30°·3 (53)}	{29°·2 (25)}	—	{30°·4 (64)}
θ_2	11°·9 (36)	—	—	{11°·2 (53)}	{12°·2 (25)}	—	{11°·0 (64)}
$P\angle$	84°·4 (37)	—	—	85°·2 (55)	85°·6 (28)	—	84°·2 (65)

* The means of these series calculated by Dr Morant have not been previously published.

† The indices and angles in curled brackets were obtained from the means of the component lengths instead of from individual values.

‡ $O_1 = 40·8$ (36).

TABLE II.

Mean Female Measurements of the Sedment and related Series.*

Characters	9th Dynasty Egyptians: Sedment (Woo)	18th—20th Dynasty Egyptians†: Thebes (Stahr)	Modern Egyptians†: Cairo (Schmidt)	Modern Abyssinians†: Tigre district (Sergi)	6th—12th Dynasty Egyptians†: Denderah (Thomson and MacIver)	12th—15th Dynasty Egyptians†: Hou & Abydos (Thomson and MacIver)	18th Dynasty Egyptians†: Abydos (Thomson and MacIver)
C	1252·5 (25)	1284·8 (43)	1211·4 (26)	1318·4 (23)	—	—	—
F	173·1 (30)	—	—	—	174·6 (149)	174·9 (88)	176·8 (67)
L	172·9 (30)	176·2 (48)	171·2 (26)	175·1 (24)	174·8 (149)	175·3 (88)	177·5 (67)
B	133·5 (30)	133·2 (44)	131·3 (26)	129·4 (24)	130·3 (147)	131·4 (88)	134·5 (67)
B'	87·4 (30)	90·4 (48)	91·7 (26)	89·8 (24)	—	—	—
H	131·7 (30)	—	128·8 (26)	—	—	—	—
H'	130·5 (30)	127·9 (43)	126·7 (26)	126·5 (22)	129·5 (147)	127·7 (88)	128·1 (66)
OH	110·4 (30)	109·2 (45)	—	107·2 (23)	—	—	—
LB	94·7 (30)	97·1 (46)	96·0 (26)	94·9 (22)	96·3 (141)	95·0 (88)	95·8 (65)
Q'	301·5 (30)	299·5 (44)	296·1 (26)	295·8 (23)	—	—	—
S	357·1 (27)	354·9 (41)	351·6 (26)	355·4 (22)	—	—	—
S_1	123·2 (30)	121·0 (42)	—	123·6 (24)	—	—	—
S_2	124·7 (30)	124·3 (42)	—	122·5 (24)	—	—	—
S_3	111·1 (27)	110·0 (45)	—	110·0 (22)	—	—	—
S_1'	107·8 (30)	106·5 (42)	—	106·7 (24)	—	—	—
S_2'	110·8 (30)	111·2 (42)	—	110·0 (24)	—	—	—
S_3'	95·4 (27)	93·8 (45)	91·4 (26)	93·0 (22)	—	—	—
U	490·4 (30)	494·2 (46)	484·9 (26)	489·3 (24)	—	—	—
G'H	67·1 (29)	66·3 (46)	63·5 (26)	65·5 (24)	66·4 (136)	67·3 (88)	67·5 (66)
GB	89·4 (28)	91·7 (43)	92·9 (26)	91·0 (20)	—	—	—
J	117·3 (25)	121·2 (43)	120·6 (26)	116·3 (13)	118.3 (121)	118·8 (82)	121·4 (64)
NH'	48·4 (29)	48·6 (45)	45·3 (26)	47·5 (24)	48·5 (138)	48·4 (87)	49·2 (67)
NB	23·6 (29)	24·3 (43)	24·6 (26)	24·5 (24)	24·9 (137)	24·4 (87)	25·1 (67)
DC	19·7 (24)	21·1 (42)	—	20·5 (23)	—	—	—
SC	10·0 (29)	10·6 (39)	—	9·3 (23)	—	—	—
$O_1(R)$	40·4 (29)	—	—	—	—	—	—
$O_1(L)$	39·8 (29)	—	—	—	—	—	—
$O_1'(R)$	37·4 (26)	37·1 (45)	—	38·5 (20)	—	—	—
$O_2(R)$	32·7 (30)	33·0 (46)	32·5 (26)	32·7 (23)	—	—	—
$O_2(L)$	32·7 (29)	—	—	—	—	—	—
G_1	47·4 (23)	—	—	—	—	—	—
G_1'	43·5 (24)	—	46·0 (24)	—	—	—	—
G_2	37·9 (24)	48·0 (39)	38·7 (24)	38·0 (23)	—	—	—
GL	89·7 (29)	92·3 (44)	95·0 (26)	91·1 (22)	92·4 (138)	90·3 (88)	91·7 (65)
fml	33·7 (27)	34·1 (43)	33·5 (26)	34·2 (21)	—	—	—
fmb	28·4 (28)	28·7 (45)	27·0 (26)	27·9 (21)	—	—	—
100 *B*/*L*	77·2 (30)	75·5 (44)	{76·7 (26)}	74·0 (24)	{74·5 (147)}	{75·0 (88)}	{75·8 (67)}
100 *H'*/*L*	75·5 (30)	72·6 (39)	{74·0 (26)}	72·5 (22)	{74·1 (147)}	{72·8 (88)}	{72·2 (66)}
100 *B*/*H'*	102·3 (30)	{104·1 (43)}‡	{103·6 (26)}	{102·3 (22)}	{100·6 (147)}	{102·9 (88)}	{105·0 (66)}
100 (*B* − *H'*)/*L*	+1·7 (30)	{+3·0 (39)}	{+2·7 (26)}	{+1·7 (22)}	{+0·5 (147)}	{+2·1 (88)}	{+3·6 (66)}
100 *G'H*/*GB*	74·9 (28)	71·9 (37)	68·5 (26)	71·8 (20)	—	—	—
100 *NB*/*NH'*	48·9 (29)	50·3 (40)	54·6 (26)	51·7 (24)	{51·3 (137)}	{50·4 (87)}	{51·0 (67)}
100 $O_2/O_1(R)$	81·2 (29)	—	—	—	—	—	—
100 $O_2/O_1(L)$	82·2 (29)	—	—	—	—	—	—
100 $O_2/O_1'(R)$	87·3 (26)	88·7 (45)	—	85·3 (20)	—	—	—
100 G_2/G_1	80·2 (18)	—	—	—	—	—	—
100 G_2/G_1'	87·7 (19)	—	84·4 (24)	—	—	—	—
100 *fmb*/*fml*	84·4 (25)	{84·2 (43)}	81·0 (26)	{81·6 (21)}	—	—	—
Oc. I.	63·5 (27)	63·4 (45)	—	61·7 (22)	—	—	—
N∠	64°·5 (29)	{65°·7 (44)}	{69°·6 (26)}	{66°·2 (22)}	{66°·1 (136)}	{64°·9 (88)}	{65°·3 (65)}
A∠	72°·9 (29)	{73°·2 (44)}	{71°·5 (26)}	{72°·4 (22)}	{72°·7 (136)}	{72°·4 (88)}	{72°·5 (65)}
B∠	42°·6 (29)	{41°·1 (44)}	{38°·9 (26)}	{41°·4 (22)}	{41°·2 (136)}	{42°·7 (88)}	{42°·2 (65)}
θ_1	30°·8 (28)	{28°·9 (44)}	—	{28°·9 (22)}	—	—	—
θ_2	11°·6 (28)	{12°·2 (44)}	—	{12°·5 (22)}	—	—	—
P∠	84°·4 (28)	85°·4 (45)	—	84°·9 (22)	—	—	—

* The ♀ mean measurements of the El-Kubanieh North and South series compared in Table III are given in *Biometrika*, Vol. XIX. (1927), Table II facing p. 117.

† The means of these series have not been previously published.

‡ The indices and angles in curled brackets were obtained from the means of the component lengths instead of from individual values.

samples which happen to be available, and some method is needed to eliminate this factor as we wish to obtain, as far as possible, a measure of the absolute divergence of the types compared which does not depend on the numbers of skulls. The correction needed has been given by Professor Pearson in a recent paper*.

* "Note on Standardisation of Method of using the Coefficient of Racial Likeness." *Biometrika*, Vol. xx^{B}. (1928), pp. 376—378. [I am by no means satisfied with this so-called correction or reduction, but was urged to it by the criticisms of Dr Morant and Professor Mahalanobis as possibly the best *at present* available. The coefficient of racial likeness as originally proposed by me was a measure by which we might *roughly* judge the likelihood that two cranial series were samples of the same population. I, personally, should lay no weight on any "crude" coefficient of the order three or less obtained from not more than 50 crania in each series as indicating a racial difference. Crania from two adjacent London burial places of the same period will give coefficients of this order. Such coefficients may easily arise from the differences in personal equation of two measurers, or from differences of class or environment of the two populations under consideration. We have also to remember the neglected correlation terms of our expression for the C.R.L. It is not alone the error of random sampling which has to be weighed.

Suppose three series A, B, C, then if the C.R.L. of A and C be much larger than that of A and B we are justified in supposing that *to the extent of the information derivable from the given data* (and beyond that we cannot get) C is racially more remote from A, than B is. To this Dr Morant and Professor Mahalanobis reply: "Yes, but the C.R.L. increases when the numbers of one or both series of crania increase, and therefore it is dependent on the numbers used, as well as on racial divergence." To illustrate this I take the 18 cases of Mr Woo's Table III and rank them (a) in average number of crania measured, (b) in order of crude coefficients, (c) in order of reduced or corrected coefficients. If ρ be the correlation of ranks we have:

$$\rho_{ab} = \cdot 7832, \quad \rho_{ac} = \cdot 2239,$$

and accordingly for the variates approximately:

$$r_{ab} = \cdot 7972, \quad r_{ac} = \cdot 2339.$$

The reduction has therefore very sensibly reduced the correlation between the numbers of crania used and the magnitude of the coefficient. Thus far the argument in favour of the reduction seems valid, but we have to consider how the changes in rank which lessen the correlation between the C.R.L.'s and the numbers of crania used have been achieved.

The series with the lowest numbers are: 18th Dynasty Egyptians (23·9), Bronze Age Spanish (24·2), Middle Dynastic Egyptians (33·1), 1st Dynasty Egyptians (33·6) and 4th—5th Dynasty Egyptians (39·9), giving the "crude" coefficients 2·74, 4·58, 2·74, 6·24 and 1·95 respectively. These occur in the ranking of the crude coefficients in the 3rd, 9th, 2nd, 12th and 1st places. In ranking of the reduced coefficients they occupy the 8th, 15th, 3rd, 17th and 1st places respectively. There is thus for the series with the fewest crania an average change of 3·4 places in the ranking. If we take the five series with the largest numbers of crania, i.e. 26th—30th Dynasty Egyptians (885·4), Modern Portuguese (494·0), 17th and 18th century Maltese (439·4), the 6th—12th Dynasty Egyptians (168·6), and the Naqada series (66·1), their ranks in the crude coefficients are 17th, 16th, 18th, 14th and 15th and these obtain with the reduced coefficients the ranks 14th, 10th, 16th, 9th and 18th or an average loss of 3·8 places. If we take the 6th—9th ranks in numbers, we find without regard to sign an average change of *one* place only, and if we take the 10th to the 13th as the next highest series of numbers the average displacement in rank is again *one*. Hence it would appear that, not only as we should expect are the series with lowest and highest numbers most changed in magnitude of coefficient by the correction, but it is these series which are also most changed in their ranking. It is, however, in the values for the high numbers of crania in the crude coefficients that we are most sure of the order of significance, and it is the coefficients with the least frequency in the data, of which we are most uncertain, which are brought into high places in the series. The average number of crania compared with the Sedment series is here about 147. Mr Woo reduces to 100. I chose 75 because it seemed a fair average of the series then before me and it is desirable to raise the reduced coefficients as little as possible above the crude. The present note is only one of warning. What counsel can we draw from it? I think it is: Be moderate in the emphasis laid on the order of "reduced" coefficients, and lay no, or very little, emphasis on series with coefficients

The coefficient is first calculated by the formula above, this being known as the crude coefficient, and it is then reduced to the value it would have if all the means of both series in the comparison were based on 100 individuals. This reduced value is:

$$50 \times \frac{\overline{n}_s + \overline{n}'_s}{\overline{n}_s \times \overline{n}'_s}\left[\frac{1}{m}\, S(a) - 1 + \frac{1}{m} \pm \cdot 67449\sqrt{\frac{2}{m}}\right].$$

Both crude and reduced coefficients are given below for all the comparisons made.

In a paper published in this journal in 1925 Dr Morant gave the mean measurements of a considerable number of early Egyptian series calculated from the data provided by a number of different anthropologists*. He suggested that two extreme forms of skull could be distinguished—known respectively as the Upper and Lower Egyptian types—but these extremes could be linked together by a number of intermediate types so that all the material then available formed a fairly continuous series. A close resemblance could be found between every sample and one, or more, of the other samples. A map given (*loc. cit.* p. 3) showed the distribution of the sites from which the series were obtained. The majority of them are in Upper Egypt and these range from Early Predynastic to Roman times. There are only four from Lower Egypt, ranging from the 4th Dynasty to the Ptolemaic Period. Sedment is more southerly than any other of the sites in Lower Egypt, but it is nearly 130 miles from the most northerly of the Upper Egyptian sites (Qau). I endeavoured to find the closest relationships, i.e. the lowest reduced coefficients of racial likeness, between the Sedment series and any others available, whether Egyptian or not. In dealing with European and Egyptian types it has been repeatedly observed that the most significant differences are almost always found for the length, breadth and height of the calvaria and the three major indices derived from these chords. Facial measurements are generally far less differentiated. By taking this fact into account, it is possible to select at once the series which will certainly not provide low coefficients of racial likeness. The following arbitrarily fixed ranges were suggested by Dr Morant and they were derived from his study of the inter-relationships of 39 European and 2 Egyptian cranial series†.

	L	*B*	*H'*	100 *B*/*L*	100 *H'*/*L*	100 *B*/*H'*
Sedment ♂ Mean	181·9	138·3	137·4	76·1	75·5	100·9
Ranges	175·4—188·4	131·8—144·8	132·9—141·9	71·6—80·6	72·0—79·0	95·9—105·9

below three, or on those containing fewer than 40 to 50 crania. Above all it is desirable on the earliest possible opportunity to take numerous samples of 20 to 60 individuals from a long series of skulls like that of the 26th—30th Dynasty Egyptians, and find their coefficients of racial likeness with each other and with the entire parent population. It would be a long task, but would throw much light on present difficulties. Editor.]

* "A Study of Egyptian Craniology from Prehistoric to Roman Times." *Biometrika*, Vol. xvii. (1925), pp. 1—52.

† "A Preliminary Classification of European Races based on Cranial Measurements." *Ibid.* Vol. xx^B. (1928), pp. 301—375.

TABLE III.

TABLE III.

Coefficients of Racial Likeness with the Male and Female Series from Sedment.

Races compared	9th Dynasty Egyptians (Sedment)									
	Crude Coefficients						Reduced Coefficients			
	Males ($\bar{n}_s$=37·1)			Females ($\bar{n}_s$=28·0)			Males		Females	
	$\bar{n}'_s$	All Characters	Indices and Angles	$\bar{n}'_s$	All Characters	Indices and Angles	All Characters	Indices and Angles	All Characters	Indices and Angles
4th and 5th Dynasty Egyptians: Deshasheh and Medum (Thomson and MacIver) ...	39·9	1·95 ± ·25 (14)	3·64 ± ·39 (6)	—	—	—	5·07	9·47	—	—
Modern Cretans (von Luschan)	47·9	2·84 ± ·25 (15)	2·72 ± ·39 (6)	—	—	—	6·79	6·51	—	—
Middle Dynastic Egyptians: El-Kubanieh North (Toldt)	33·1	2·74 ± ·18 (28)	2·72 ± ·29 (11)	18·6	0·92 ± ·18 (28)	3·11 ± ·29 (11)	7·83	7·77	4·10	13·87
18th—20th Dynasty Egyptians: Thebes (Stahr)	52·8	3·64 ± ·18 (29)	6·18 ± ·29 (11)	43·7	2·69 ± ·18 (29)	3·34 ± ·29 (11)	8·38	14·18	7·88	9·79
Modern Egyptians: Cairo (Schmidt) ...	44·9	3·41 ± ·18 (27)	2·62 ± ·32 (9)	25·8	7·21 ± ·18 (27)	11·10 ± ·32 (9)	8·39	6·45	26·85	41·33
Modern Abyssinians: Tigre District (Sergi)	61·8	4·52 ± ·18 (28)	5·68 ± ·29 (11)	22·1	3·03 ± ·18 (28)	4·60 ± ·29 (11)	9·17	12·25	12·27	18·62
Early and Middle Dynastic Egyptians: El-Kubanieh South (Toldt)	63·5	4·35 ± ·18 (28)	5·49 ± ·29 (11)	42·6	4·03 ± ·18 (28)	6·23 ± ·29 (11)	9·29	11·72	11·88	18·36
18th Dynasty Egyptians: Thebes? (Schultz and Oetteking)	23·9	2·74 ± ·17 (31)	4·05 ± ·28 (12)	—	—	—	9·43	13·93	—	—
6th—12th Dynasty Egyptians: Denderah (Thomson and MacIver)	168·6	7·45 ± ·25 (14)	8·69 ± ·39 (6)	140·4	5·74 ± ·24 (14)	8·08 ± ·39 (6)	12·25	14·29	12·29	17·31
Modern Portuguese (Ferraz de Macedo) ...	494·0	8·93 ± ·22 (18)	16·36 ± ·43 (5)	—	—	—	12·94	23·70	—	—
18th Dynasty Egyptians: Abydos (Thomson and MacIver)	50·0	5·59 ± ·25 (14)	7·57 ± ·39 (6)	66.1	7·24 ± ·25 (14)	7·61 ± ·39 (6)	13·12	17·77	18·40	19·35
Canary Islands Modern (Lajard)	50·0	5·62 ± ·22 (18)	3·85 ± ·43 (5)	50·0	8·14 ± ·22 (18)	4·51 ± ·43 (5)	13·19	9·04	22·68	12·56
12th—15th Dynasty Egyptians: Hou and Abydos (Thomson and MacIver) ...	65·9	6·29 ± ·25 (14)	8·55 ± ·39 (6)	87·4	3·76 ± ·25 (14)	6·82 ± ·39 (6)	13·25	18·01	8·87	16·08
26th—30th Dynasty Egyptians: Gizeh (Davin and Pearson)	885·4	10·11 ± ·17 (31)	15·72 ± ·28 (12)	567·3	5·83 ± ·17 (31)	4·45 ± ·28 (12)	14·20	22·07	10·92	8·34
Bronze Age Spanish (Jacques)	24·2	4·58 ± ·21 (21)	6·89 ± ·39 (6)	33·8	10·11 ± ·21 (21)	12·36 ± ·39 (6)	15·64	23·52	33·01	40·36
17th and 18th century Maltese (Buxton) ...	439·4	11·11 ± ·21 (20)	6·34 ± ·34 (8)	—	—	—	16·24	9·27	—	—
1st Dynasty Egyptians, Private Tombs: Abydos (Thomson and MacIver) ...	33·6	6·24 ± ·25 (14)	8·27 ± ·39 (6)	55·1	10·44 ± ·25 (14)	13·84 ± ·39 (6)	17·70	23·45	27·35	36·26
Predynastic Egyptians: Naqada *A* and *Q* series (Fawcett)	66·1	8·46 ± ·17 (31)	12·50 ± ·28 (12)	109·5	7·92 ± ·19 (25)	12·75 ± ·32 (9)	17·80	26·30	17·76	28·59

If the ♂ mean for any series falls outside the range indicated for any one, or more, of the 6 characters then it will be safe to assume that there is no close relationship between it and the Sedment series. Comparison was made in this way with about 80 ♂ European and 24 ♂ ancient and modern Egyptian types. All except 23 could be excluded on account of the aberrance of one or other of the 6 measurements compared, and it was only thought necessary to calculate the coefficients of racial likeness with ten predynastic and dynastic Egyptian, three modern Egyptian, eight European, one Abyssinian and one Canary Island series. The reduced values of these twenty-three coefficients range from 5·07 to 42·72 and the eighteen values less than 20 are arranged in order in Table III. We may feel confident that comparison is made there with all the series available which are most closely related to the Sedment one. All the coefficients which could be calculated with the ♀ Sedment means are also in Table III. Male means of the most closely related series are in Table I and female means in Table II*. Comparisons are made in Table III between the ♂ Sedment series and 18 others. The coefficients of racial likeness are given for the standard set of 31 characters, or as many of them as

* The comparative material used in the present paper was derived from the following sources:

(i) Thomson, A. and Randall-MacIver, D.: *The Ancient Races of the Thebaid.* Oxford (1905). Use is made of five series for which individual measurements are given in this work. The ♂ means of these and of all the other ancient Egyptian and the Abyssinian series are given in *Biometrika*, Vol. XVII. (1925), pp. 14—36.

(ii) Stahr, Hermann: *Die Rassenfrage im antiken Aegypten. Kraniologische Untersuchungen an Mumienköpfen aus Theben.* Leipzig (1907).

(iii) Toldt, C.: "Anthropologische Untersuchung der menschlichen Ueberreste aus den altägyptischen Gräberfeldern von El-Kubanieh." *Denkschriften der Akademie der Wissenschaften in Wien, Math.-naturwiss. Klasse*, Band XCVI. (1919), S. 593—672.

(iv) Schmidt, Emil: *Die anthropologischen Sammlungen Deutschlands.* Leipzig Catalogue (1887), also published with *Archiv für Anthropologie*, Band XVII.

(v) Sergi, Sergio: *Crania Habessinica. Contributo all' Antropologia dell' Africa Orientale.* Rome (1912).

(vi) Oetteking, B.: "Kraniologische Studien an Altägypten." *Archiv für Anthropologie*, Band XXXVI. (1909), S. 1—90. The sexes of the skulls used in computing the ♂ means are given by Schultz in *ibid.* Band XLIV. (1918), S. 72—79.

(vii) Pearson, Karl and Davin, Adelaide G.: "On the Biometric Constants of the Human Skull." *Biometrika*, Vol. XVI. (1924), pp. 328—363.

(viii) Fawcett, Cicely D.: "A Second Study of the Variation and Correlation of the Human Skull, with special Reference to the Naqada Crania." *Ibid.* Vol. I. (1902), pp. 408—467.

(ix) v. Luschan, Felix: "Beiträge zur Anthropologie von Kreta." *Zeitschrift für Ethnologie*, Jahrgang XIV. (1913), S. 307—393.

(x) Lajard: "La race Ibère. Crânes des Canaries et des Açores." *Bulletins de la Société d'Anthropologie de Paris*, IV^e^ série, tome III. (1892), pp. 294—326.

(xi) de Macedo, Francisco Ferraz: *Crime et Criminel. Essai de synthétique d'observations anatomiques, physiologiques, pathologiques et psychiques sur les délinquants et morts selon la méthode et les procédés anthropologiques les plus rigoureux.* Lisbon, 1892.

(xii) Buxton, L. H. Dudley: "The Ethnology of Malta and Gozo." *Journal of the Royal Anthropological Institute*, Vol. LII. (1922), pp. 164—211.

(xiii) Siret, H. and L.: *Les premiers âges du métal dans le sud-est de l'Espagne.* Antwerp (1887). The skull measurements are given in the section on *Ethnologie* (pp. 267—396) by V. Jacques. The means are for the pooled Argar, Gerúndia and Puerto-Blanco series. The ♂ means have been given already in *Biometrika*, Vol. XX^B^. (1928), pp. 374 and 375.

could be used in a particular comparison, and for the 12, or fewer, indices and angles which form part of the total 31. The orders given by the two kinds of coefficients are not in good agreement and this may be supposed due to the fact that the values for indices and angles are not all based on the same number of characters. It is known that the average contributions of the individual characters to the coefficients are far from being constant and the omission of a few measurements may have a marked effect if the total number compared is small. Female data for 13 of the 18 series are available and the corresponding ♂ and ♀ reduced coefficients are in fair agreement with one another. The Sedment and several of the other series are short ones and the errors of random sampling are probably large enough to account for most of the sexual differences of this kind observed. The most marked divergence is found in the comparisons with the modern Egyptian series from Cairo, the ♂ reduced coefficient being 8·39 and the ♀ 26·85. The disagreement in this case is almost certainly due to the fact that the ♂ and ♀ samples from Cairo do not represent the same racial population*. The mean indices and angles for the two sexes (cf. Tables I and II) have differences which are too large to be attributed to random sampling. The ♂ and ♀ mean indices and angles of the Sedment series are in close agreement and there is every reason to believe that they represent the same race. The relationships of that type will be measured most accurately by the ♂ reduced coefficients of racial likeness for all characters given in Table III. It is extremely satisfactory to find that the closest connection is with the 4th and 5th dynasty series from Deshasheh and Medum, since Sedment is closer to these two than to any of the other Egyptian sites represented, and the time interval between the series is also small. The reduced coefficient of 5·07 indicates a very close degree of resemblance and it will be of interest to compare it with the closest connections which have been found for other Egyptian series. In Morant's paper all the lowest crude coefficients of racial likeness are given for 23 closely related ♂ series, omitting the Aeneolithic Egyptian one which is of an aberrant type. These are all Egyptian of periods ranging from early predynastic to Roman times except in the Abyssinian series (Tigre district) which is modern. The coefficients were reduced and the distribution of the lowest values for each series is given below.

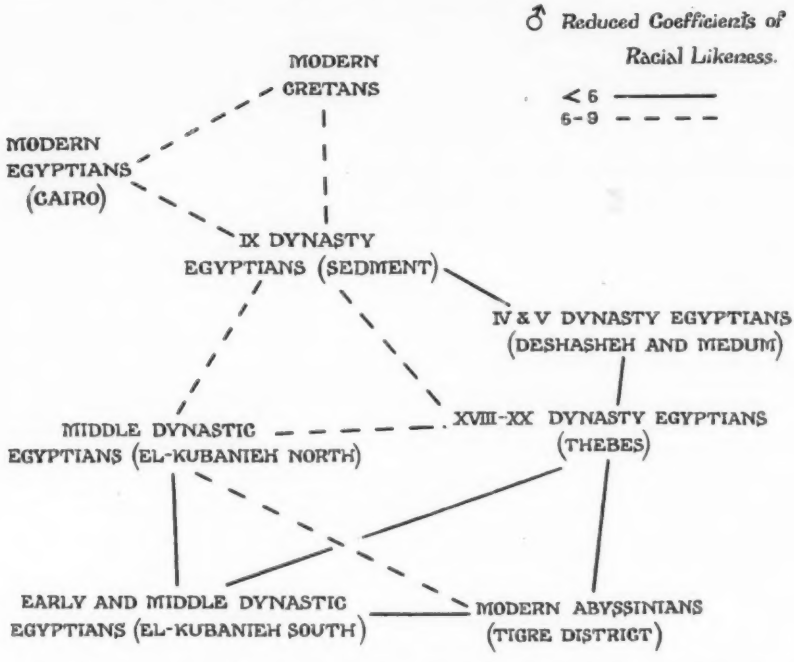
	−0·5—0·5	*0·5—1·5*	*1·5—2·5*	*2·5—3·5*	*3·5—4·5*
Lowest Reduced Coefficients	2	10	7	3	1

The lowest ♂ reduced coefficient in Table III is 5·07, so the Sedment series must be supposed less typically Egyptian than any of the 23 previously described. The fact that its second closest connection is with Modern Cretans again suggests foreign admixture. Seven series of the Egyptian type, of which one is a modern

* The ♂ and ♀ indices and angles of this series are in bad agreement. Judging from the nasal index and angle the ♀ series is predominantly of Negroid origin; though the same is not true of the ♂ series.

series from Cairo and another is of modern Abyssinians, follow, then the Portuguese, and other Egyptian, North African and European types. The only European populations represented are ones from the Mediterranean area. The feature of the relationships observed which is most unexpected is the fact that the Sedment sample bears a closer resemblance to modern Cretans than to all except one of the available Egyptian types. To throw further light on the point the coefficients of racial likeness were computed between all pairs of the seven series which were found to be most closely related to the one from Sedment. Crude and reduced

RELATIONSHIPS OF SERIES RESEMBLING THE SEDMENT SERIES.


♂ Reduced Coefficients of Racial Likeness.
< 6
6-9
MODERN CRETANS
MODERN EGYPTIANS (CAIRO)
IX DYNASTY EGYPTIANS (SEDMENT)
IV & V DYNASTY EGYPTIANS (DESHASHEH AND MEDUM)
MIDDLE DYNASTIC EGYPTIANS (EL-KUBANIEH NORTH)
XVIII-XX DYNASTY EGYPTIANS (THEBES)
EARLY AND MIDDLE DYNASTIC EGYPTIANS (EL-KUBANIEH SOUTH)
MODERN ABYSSINIANS (TIGRE DISTRICT)

values are given in Table IV, and the diagram below shows all the closest connections. The arrangement given by these means corresponds closely with the geographical distribution of the types since Sedment, Deshasheh and Medum are in Lower Egypt, while El-Kubanieh is south of Thebes. The Sedment series appears to be distinguished from all other early Egyptian ones by bearing a peculiarly close resemblance to both modern Cretans and modern Egyptians from Cairo. The earlier and later populations of Lower Egypt are represented by the 4th and 5th dynasty series from Deshasheh and Medum and by the 26th—30th dynasty series from Gizeh. Neither of these is so closely related to the modern

Coefficients of Racial Likeness between Series closely related to the Sedment Series (Male and Female).

			4th and 5th Dynasty Egyptians: Deshasheh and Medum	Modern Cretans	18th—20th Dynasty Egyptians: Thebes	Middle Dynastic Egyptians: El-Kubanieh North	Modern Egyptians: Cairo	Modern Abyssinians: Tigre District	Early and Middle Dynastic Egyptians: El-Kubanieh South
Crude Coefficients	4th and 5th Dynasty Egyptians: Deshasheh and Medum	♂ ♀	39·9 —	— —	6·25 ± ·25 (14) —	1·16 ± ·25 (14) —	4·94 ± ·25 (14) —	9·80 ± ·25 (14) —	5·05 ± ·25 (14) —	5·96 ± ·25 (14) —
	Modern Cretans	♂ ♀	47·9 —	6·25 ± ·25 (14) —	— —	7·72 ± ·25 (15) —	10·11 ± ·25 (15) —	4·00 ± ·25 (15) —	13·13 ± ·25 (15) —	16·33 ± ·25 (15) —
	18th—20th Dynasty Egyptians: Thebes	♂ ♀	52·8 43·7	1·16 ± ·25 (14) —	7·72 ± ·25 (15) —	— —	3·27 ± ·18 (27) 1·14 ± ·18 (27)	6·76 ± ·19 (25) 5·96 ± ·19 (25)	2·58 ± ·18 (28) 1·96 ± ·18 (28)	3·18 ± ·18 (27) 2·59 ± ·18 (27)
	Middle Dynastic Egyptians: El-Kubanieh North	♂ ♀	33·1 18·6	4·94 ± ·25 (14) —	10·11 ± ·25 (15) —	3·27 ± ·18 (27) 1·14 ± ·18 (27)	— —	5·37 ± ·19 (24) 4·69 ± ·19 (24)	3·16 ± ·18 (27) 0·99 ± ·19 (26)	1·64 ± ·18 (28) 0·56 ± ·18 (28)
	Modern Egyptians: Cairo	♂ ♀	44·9 25·8	9·80 ± ·25 (14) —	4·00 ± ·25 (15) —	6·76 ± ·19 (25) 5·96 ± ·19 (25)	5·37 ± ·19 (24) 4·69 ± ·19 (24)	— —	8·45 ± ·19 (24) 2·64 ± ·19 (24)	6·20 ± ·19 (24) 5·24 ± ·19 (24)
	Modern Abyssinians: Tigre District	♂ ♀	61·8 22·1	5·05 ± ·25 (14) —	13·13 ± ·25 (15) —	2·58 ± ·18 (28) 1·96 ± ·18 (28)	3·16 ± ·18 (27) 0·99 ± ·19 (26)	8·45 ± ·19 (24) 2·64 ± ·19 (24)	— —	1·91 ± ·20 (23) 1·91 ± ·19 (26)
	Early and Middle Dynastic Egyptians: El-Kubanieh South	♂ ♀	63·5 42·6	5·96 ± ·25 (14) —	16·33 ± ·25 (15) —	3·18 ± ·18 (27) 2·59 ± ·18 (27)	1·64 ± ·18 (28) 0·56 ± ·18 (28)	6·20 ± ·19 (24) 5·24 ± ·19 (24)	1·91 ± ·20 (23) 1·91 ± ·19 (26)	— —
Reduced Coefficients	4th and 5th Dynasty Egyptians: Deshasheh and Medum	♂ ♀	— —	14·46 —	2·54 —	11·56 —	23·12 —	10·41 —	12·24 —	
	Modern Cretans	♂ ♀	14·46 —	— —	15·25 —	25·69 —	8·55 —	24·08 —	29·86 —	
	18th—20th Dynasty Egyptians: Thebes	♂ ♀	2·54 —	15·25 —	— —	8·05 4·35	13·82 18·36	4·50 7·17	5·48 4·67	
	Middle Dynastic Egyptians: El-Kubanieh North	♂ ♀	11·56 —	25·69 —	8·05 4·35	— —	14·15 21·73	7·48 4·90	3·77 2·17	
	Modern Egyptians: Cairo	♂ ♀	23·12 —	8·55 —	13·82 18·36	14·15 21·73	— —	16·15 11·07	11·68 16·24	
	Modern Abyssinians: Tigre District	♂ ♀	10·41 —	24·08 —	4·50 7·17	7·48 4·90	16·15 11·07	— —	3·05 6·57	
	Early and Middle Dynastic Egyptians: El-Kubanieh South	♂ ♀	12·24 —	29·86 —	5·48 4·67	3·77 2·17	11·68 16·24	3·05 6·57	— —	

Cretans. The ranges of the coefficients on which the diagram above is based are for ♂ readings. There are no adequate ♀ means for modern Cretans and the 4th and 5th Dynasty Egyptians from Deshasheh and Medum. The ♀ coefficients between the remaining six series furnish a scheme of relationship which is closely similar to the one already considered except in one particular; the Sedment and modern series from Cairo have ♀ means which are much further removed than the corresponding ♂ means.

TABLE V.

Values of "a" between the Male and Female Sedment and other Series.

Races	4th and 5th Dynasty Egyptians: Deshasheh and Medum (Thomson and MacIver)	Modern Cretans (von Luschan)	18th—20th Dynasty Egyptians: Thebes (Stahr)	Middle Dynastic Egyptians: El-Kubanieh North (Toldt)	Mean a for 18 Series (No. of a's)	18th—20th Dynasty Egyptians: Thebes (Stahr)	Middle Dynastic Egyptians: El-Kubanieh North (Toldt)	Mean a for 13 Series (No. of a's)
Sex	♂	♂	♂	♂	♂	♀	♀	♀
$\bar{n}$	39·9	47·9	52·8	33·1	—	43·7	18·6	—
$100\,H'/L$	21·01	1·62	18·53	1·35	17·85 (18)	17·57	3·66	16·14 (13)
Oc. I.	—	—	22·35	4·56	16·28 (7)	2·38	7·73	6·44 (5)
$100\,B/L$	1·51	6·22	5·40	15·42	15·37 (18)	7·81	12·02	15·03 (13)
Q'	—	—	·45	—	12·99 (6)	·91	—	4·07 (5)
O_1' (or O_1)	—	—	15·44	16·07	11·95 (9)	·61	·00	5·03 (7)
B'	—	16·89	7·18	5·67	11·38 (13)	11·54	3·88	22·94 (9)
G_2	—	—	·43	—	11·11 (8)	·03	—	1·29 (6)
H'	1·66	·01	12·07	1·62	9·84 (18)	6·26	·15	4·89 (13)
$100\,G_2/G_1'$ (or G_2/G_1)	—	—	—	—	8·77 (4)	—	—	1·99 (2)
$100\,G'H/GB$	—	—	4·86	3·76	8·23 (9)	7·14	1·24	10·46 (7)
B	·98	1·21	1·03	14·26	7·54 (18)	·07	3·01	5·00 (13)
$100\,O_2/O_1'$ (or O_2/O_1)	—	—	4·76	·44	7·46 (9)	1·53	·90	7·48 (7)
$100\,NB/NH'$ (or NB/NH)	·35	0·00	11·49	6·14	7·08 (18)	2·32	·06	7·42 (13)
$G'H$	·03	7·15	2·58	1·88	5·75 (16)	·80	·03	2·68 (12)
L	6·32	2·01	1·83	·06	5·19 (18)	9·03	5·13	13·46 (13)
J	0·00	5·53	1·76	1·89	5·07 (18)	12·52	·05	8·76 (13)
U	—	—	·09	3·66	4·95 (11)	1·89	·00	7·93 (9)
$100\,B/H'$	2·63	·31	3·90	4·77	4·78 (18)	3·46	·85	5·09 (13)
NB	2·35	·63	4·58	·54	4·29 (18)	3·15	·18	5·70 (13)
NH' (or NH)	·86	1·22	·95	5·62	4·18 (18)	·11	·77	2·97 (13)
O_2	—	—	1·55	5·11	3·96 (12)	·49	·56	1·98 (9)
$A\angle$	1·04	8·15	4·59	1·24	2·60 (15)	·14	·57	1·38 (11)
$N\angle$	·29	5·01	·71	·01	2·39 (15)	2·38	3·52	8·33 (11)
G_1' (or G_1)	—	—	—	·12	2·18 (6)	—	·26	6·33 (4)
fmb	—	—	·78	3·04	2·16 (11)	·40	·00	1·66 (8)
C	—	—	·93	·61	2·13 (7)	1·94	·10	1·39 (6)
$P\angle$	—	—	1·35	2·18	1·99 (7)	1·98	2·53	·94 (6)
S	—	—	·38	·14	1·63 (11)	·72	3·04	5·07 (9)
fml	—	—	1·80	2·81	1·47 (11)	·55	·36	1·10 (8)
LB	1·30	·60	1·71	·73	·95 (17)	8·44	1·84	2·72 (12)
$100\,fmb/fml$	—	—	·07	·01	·92 (11)	·02	·17	1·04 (8)

4. *A Comparison of Single Characters.* The a's found in computing the coefficients give a convenient measure of the significance of the differences between single mean measurements. The difference may be supposed to be definitely significant if the a is greater than 10. All values available for the 31 characters are given in Table V for the 4 ♂ and 2 ♀ series which most closely resemble the Sedment type. The mean a's are also given for the 18 series with which comparisons are made in Table III. As is usually found, the average contributions which the characters make to the coefficients vary greatly. Mean a's have been given based on 820 comparisons between 37 European and 4 North African series*. The value for the cephalic index is almost twice as great as that found for any other character, and next in order are B, 100 B/H', L and O_1. The characters 100 H'/L, the

TABLE VI.

Male Means of Egyptian and other Series†.

		L	B	H'	100 B/L	100 H'/L	*Oc. I.*	B'
Egyptians: 9th Dynasty (Sedment)		181·9	138·3	137·4	76·1	75·5	64·0	92·6
Modern Cretans		180·2	139·4	137·5	77·5	76·3	—	96·1
Modern Egyptians (Cairo)		176·1	137·6	134·6	78·1	76·4	—	94·3
Egyptians: 4th and 5th Dynasties (Deshasheh and Medum)		184·9	139·3	136·0	75·4	72·6	—	—
Egyptians: 18th—20th Dynasties (Thebes)		183·5	137·3	133·7	74·8	72·8	60·7	94·9
Egyptians: Middle Dynasties (El-Kubanieh N.)		182·2	134·2	135·9	73·7	74·7	61·9	90·4
Other Egyptian series: predynastic to Roman times‡	Range	181·4—185·8	130·8—139·2	130·7—135·1	71·7—76·0	71·3—73·9	59·9—61·9	91·1—96·2
	No. of Series	16	16	16	16	16	5	7

occipital index (*Oc. I.*), H' and B' are nearer the middle of the range and hardly any significant differences were found for $N\angle$, $A\angle$, NB, 100 NB/NH' and 100 fmb/fml. In the present comparison the indices 100 H'/L, *Oc. I.* and 100 B/L are the most variable characters, and these are the ones most likely to distinguish the Sedment from all the other series. The height-length index is the only character showing a significant difference in the comparison with the 4th and 5th dynasty Egyptians from Deshasheh and Medum and the minimum frontal diameter is the only one which differentiates the Cretans from the Sedment series. Mean values for some of the characters which differ most significantly are shown in Table VI. The length of the Sedment type is slightly greater than the smallest mean recorded for a

* *Biometrika*, Vol. xx[B]. (1928), Table XVI, facing p. 336.

† All the means in this table are based on 30 or more skulls.

‡ The means of these series are given in *Biometrika*, Vol. xvii. (1925), pp. 14—36 (Fouquet's measurements being excluded) and *Ibid.* Vol. xix. (1927), Table II, facing p. 117.

dynastic Egyptian series; the breadth is a millimetre less than the largest of these means and the cephalic index is 0·1 greater than any other recorded for this group of closely allied racial types. Both the basio-bregmatic height (H') and the height-length index ($100\ H'/L$) are more clearly differentiated from the continuous series given by previously measured early Egyptian samples. In the case of all five of these characters the tendency to diverge, or the actual divergence, of the Sedment constant from the inter-racial range is in the direction of the still more divergent mean for modern Cretans. The evidence of these measurements is not independent evidence, of course, since they are known to be quite highly correlated with one another both intra- and inter-racially. The high mean value of α found for the occipital index (*Oc. I.*) is seen from Table VI to have been occasioned by the fact that the Sedment value is appreciably higher than any other found for an Egyptian series. Means of this character have been given for a considerable number of racial types*. They range from 58·0 to 68·8; all the lowest values are for Western European and all the highest for African Negro races. The highest as yet found for European races are for Rumanians (62·7), Serbo-Croats (62·8), Greeks (62·9) and Turks (63·3). The occipital index is unfortunately not available for the modern Cretans. It may be suggested that the high means found for the Sedment series (♂ 64·0, ♀ 63·5) indicate Negroid admixture, but a comparison of characters which are better criteria of the Negro skull, such as angular measurements of prognathism and the nasal index, fails entirely to substantiate that view. Minimum frontal breadths (B') are also compared in Table VI and the Sedment mean falls within the range furnished by other early Egyptian types. The same has been found for every other character measured, including a number which are not used in computing the coefficient of racial likeness. The nasal breadth, index and angle, however, are almost as low as any of the other means. The distinctiveness of the Sedment type is thus seen to depend on very few of the characters which may be compared.

5. *Type Contours.* The ♂ and ♀ type contours for the Sedment series were constructed from the mean measurements of the individual contours in the way usually employed in the Biometric Laboratory†. They are given in Figs. I—VI and the means themselves will be found in Tables VII—IX. A number of chords and angular measurements of the types should agree very approximately with absolute mean readings and a check on both is thus obtained. Comparisons were made in the case of the auricular heights of the transverse section and a number of measurements taken on the sagittal figures. The maximum difference between the readings obtained by the two methods was 0·5 mm. for chords and 0°·4 for angles. This is a satisfactory agreement‡.

* *Biometrika*, Vol. XVI. (1924), pp. 334 and 335. † See *Ibid.* Vol. XIV. (1923), pp. 227—244.

‡ Perfect correspondence between all the absolute and contour measurements which are usually compared is not to be expected, even if the personal equations of both measuring and drawing were zero, owing to the methods employed in tracing individual and constructing type contours. For example, the craniophor auricular height is the *maximum* projective height from the auricular axis, but the highest point on the transverse type is obtained from the mean of the vertical heights bisecting the auricular

The transverse vertical contour is drawn through the auricular points—the "porions" of Martin—perpendicular to the Frankfurt horizontal plane. The ♂ and ♀ types (Figs. I and II) are almost symmetrical, the maximum difference between the right and left sides of the same ordinate being 0·9 mm. in favour of the right

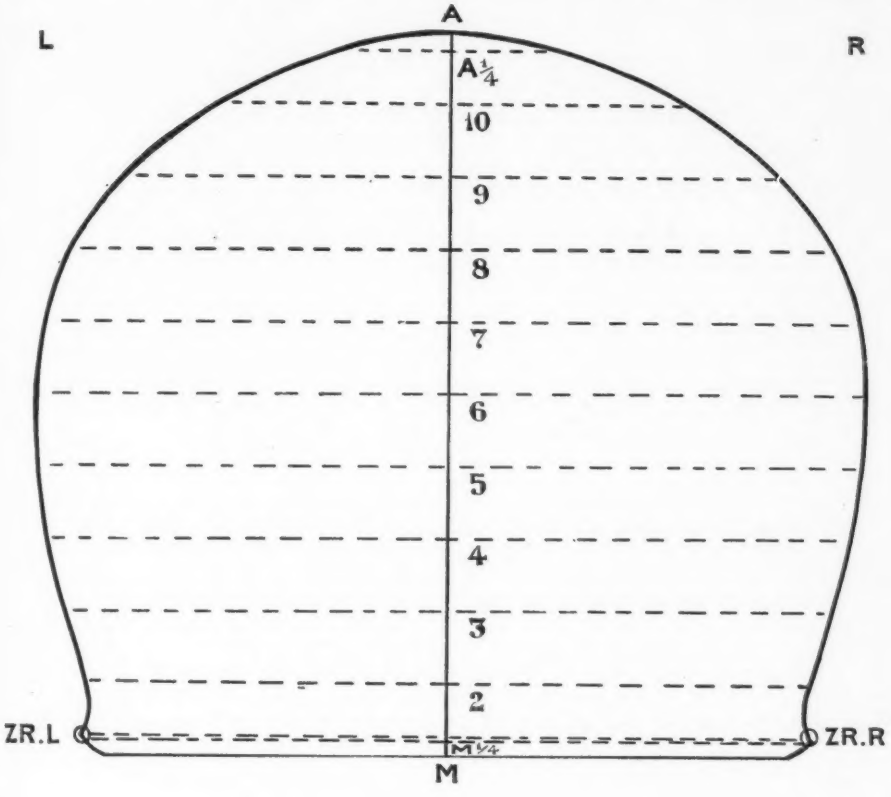


Fig. I. Transverse Type Contour of 39 ♂ Egyptian Skulls from Sedment.

side on the ♂ figure and 1·5 mm. in favour of the left side on the ♀ figure. The maximum breadths of both are close to the 6th parallel. For all racial type contours which have yet been published the maximum breadths lie between the

axis. The two may not correspond on an asymmetrical skull. The fact that the point of the tracer was in a few cases raised or lowered to pass through some standard points shown on the sagittal figures will lead to projected lengths which are slightly different from the direct calliper readings.

4th and 6th parallels, and in several cases they are between the 4th and 5th. The relative position of the major diameter appears to be a character which distinguishes the Sedment Egyptian type from all others available. The point where the line joining the most lateral points right and left meets the axis *MA* may be taken to

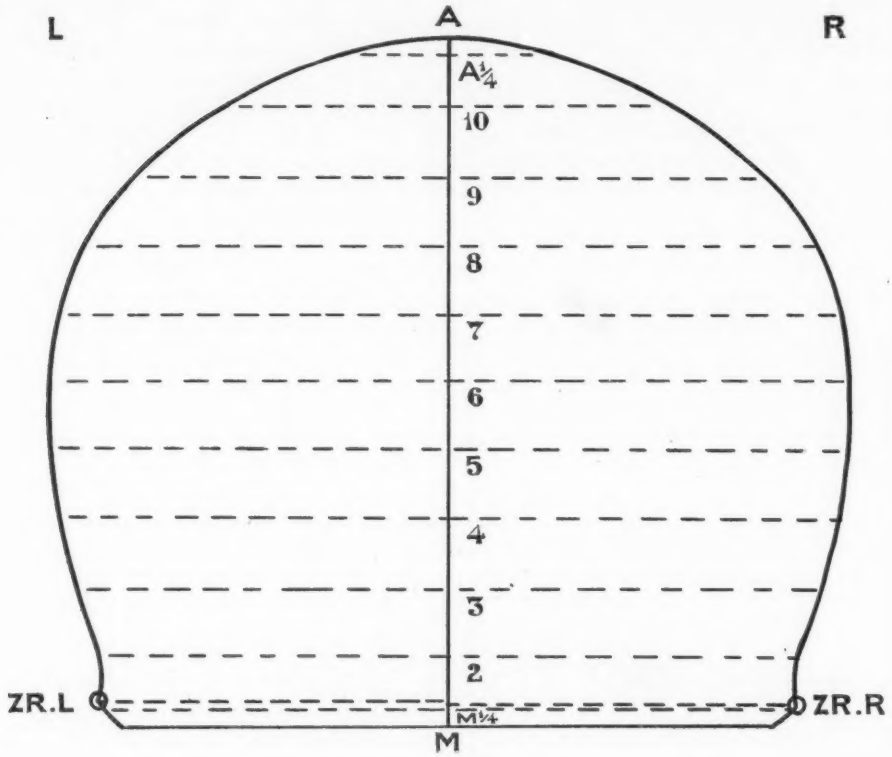


FIG. II Transverse Type Contour of 30 ♀ Egyptian Skulls from Sedment

indicate the height of the maximum horizontal diameter above the auricular axis. The character in question can be conveniently measured by expressing the distance of this point from *M* as a percentage of *MA*. The indices on the following page are given by type contours each based on 20 or more skulls; the index is for the ♂ type unless otherwise indicated.

45—50. 9th Dynasty Egyptians: Sedment 48·7 (♀ 47·4), Congo Negroes: Fernand Vaz, 1880* 45·6.

40—45. Predynastic Egyptians: Badari 44·4 (♀ 43·2)†, 26th—30th Dynasty Egyptians: Gizeh 43·3*, 1st Dynasty Egyptians: Abydos 43·1‡, Tamils 41·6§, Burmese A 41·4 (♀ 38·0)‖, Congo Negroes: Batetelu* 41·3, Northern Chinese 40·3¶, Hokien Chinese 40·1§.

35—40. Prehistoric Chinese 39·0¶, Congo Negroes: Fernand Vaz, 1864* 38·8, Basques 38·4**, Nepalese 37·5††, Tibetan A 37·3††.

30—35. 17th century English: Whitechapel 34·8*, 17th century English: Farringdon St‡‡ 34·7 (♀ 34·4), Eskimo 34·5*, Anglo-Saxons 31·8 (♀ 30·9)§§.

TABLE VII.

Mean Measurements of Transverse Vertical Contours.

Sex	Cases	*MA*	1*R*=1*L*	¼*R*	¼*L*	2*R*	2*L*	3*R*	3*L*	4*R*	4*L*
♂	39	114·9	54·1	58·2	58·2	57·8	57·7	61·3	61·0	64·5	64·1
♀	30	109·9	51·5	55·1	54·9	55·9	55·5	59·3	58·8	62·5*	61·9

Sex	Cases	5*R*	5*L*	6*R*	6*L*	7*R*	7*L*	8*R*	8*L*	9*R*	9*L*
♂	39	65·9	65·5	66·3	65·8	65·3	64·8	61·4	60·9	52·6	52·0
♀	30	63·9	63·3	63·9	63·2	62·6	61·9	58·9	58·1	50·9	49·9

Sex	Cases	10*R*	10*L*	*A*¼*R*	*A*¼*L*	*ZR*, *R*		*ZR*, *L*	
						y	*x*	*y*	*x*
♂	39	37·8	36·9	17·4	17·6	58·5	3·3	58·5	3·4
♀	30	35·4	35·4	16·3	17·8	55·6	3·2	55·6	3·6

* Number of cases = 29.

The four Egyptian series available are all among the types found with the five highest values of this index. The Western European races are at the other extreme of the range and, unfortunately, no data for Eastern European types can be given. It is interesting to observe that the ♂ index is greater than the ♀ in every case. For most races the mean auricular height is less than the mean auricular breadth, so that the index which expresses *MA* as a percentage of the parallel 1 is less than

* Benington: *Biometrika*, Vol. VIII. (1911), pp. 157—193.
† Stoessiger: *Ibid.* Vol. XIX. (1927), pp. 136 and 137.
‡ Motley: *Ibid.* Vol. XVII. (1925), p. 47.
§ Harrower: *Transactions of the Royal Society of Edinburgh*, Vol. LIV. (1926), pp. 592 and 594.
‖ Tildesley: *Biometrika*, Vol. XIII. (1921), pp. 188 and 191.
¶ Black: *Palaeontologia Sinica*, Series D, Vol. VII. (1928), p. 47.
** Morant: *Biometrika*, Vol. XXI. (1929), p. 78.
†† Morant: *Ibid.* Vol. XVI. (1924), pp. 76 and 77.
‡‡ Hooke: *Ibid.* Vol. XVIII. (1926), pp. 43 and 44.
§§ Morant: *Ibid.* Vol. XVIII. (1926), pp. 90 and 91.

100. This is so for the 26th—30th and for the 1st Dynasty Egyptian contours. The indices for the Badari Egyptians (♂ 107·9, ♀ 110·0) are the highest that have yet been found however. The value given by the ♂ Tamil type is 107·7 and the Sedment indices (♂ 106·2, ♀ 106·7) are also close to the extreme. Other measurements of the transverse type contour which have been used for comparative purposes do not distinguish our Sedment from the other series. Figs. I and II are differentiated from all others available by having their maximum breadths peculiarly high up and their heights are also large in proportion to the auricular widths. In these and other respects they bear a closer resemblance to the Badari than to the 1st or the 26th—30th Dynasty sections. Superposing the Sedment and Badari types, with the aid of the tracings provided, a close correspondence is found for both sexes. The former outlines are slightly higher and broader than the others.

TABLE VIII.

Mean Measurements of Horizontal Contours.

Sex	Cases	*FO*	*F¼R*	*F¼L*	*F½R*	*F½L*	*2R*	*2L*	*2½R*	*2½L*	*3R*	*3L*
♂	38	180·6	22·2	22·7	33·2	33·4	45·3	45·1	45·9	45·9	47·6	48·0
♀	30	172·4	21·3	21·2	32·9	32·8	44·2	43·8	44·5	44·5	47·0	47·1

Sex	Cases	*4R*	*4L*	*5R*	*5L*	*6R*	*6L*	*7R*	*7L*	*8R*	*8L*
♂	38	54·2	54·4	60·9*	60·8	65·6	65·4	67·4	67·1	65·5	65·2
♀	30	53·1	53·0	59·7†	59·3	64·5†	63·7	65·5	64·5	62·9	62·1

Sex	Cases	*9R*	*9L*	*10R*	*10L*	*O¼R*	*O¼L*	*TR*		*TL*	
								y	*x*	*y*	*x*
♂	38	59·3	58·9	46·6	45·9	28·0	26·6	48·0	20·1	47·4	19·4
♀	30	56·3	55·6	44·0	43·5	26·3	25·9	45·2	17·9	44·7	17·4

* Number of cases = 37. † Number of cases = 29.

The glabella horizontal section (Figs. III and IV) is drawn through the glabella parallel to the Frankfurt horizontal plane. The point *F* is the glabella, T_R and T_L mark the crossing of the temporal lines and *O* is the occipital point in the median sagittal plane. The type figures are almost exactly symmetrical. The maximum difference between the right and left sides of the same ordinate (see Table VIII) is 1·4 mm. in favour of the right side for the ♂ figure and 1·0 mm. in favour of the right side for the ♀. As for all other types which have yet been constructed, the maximum breadths are between the 6th and 7th parallels. The outlines have no characteristics which are very distinctive. The temporal fossae are shallow though more marked, as usual, on the ♂ than on the ♀ figure. A number of indices derived from measurements of the horizontal type contour have been used to compare degrees of frontal development and other features. The variation shown

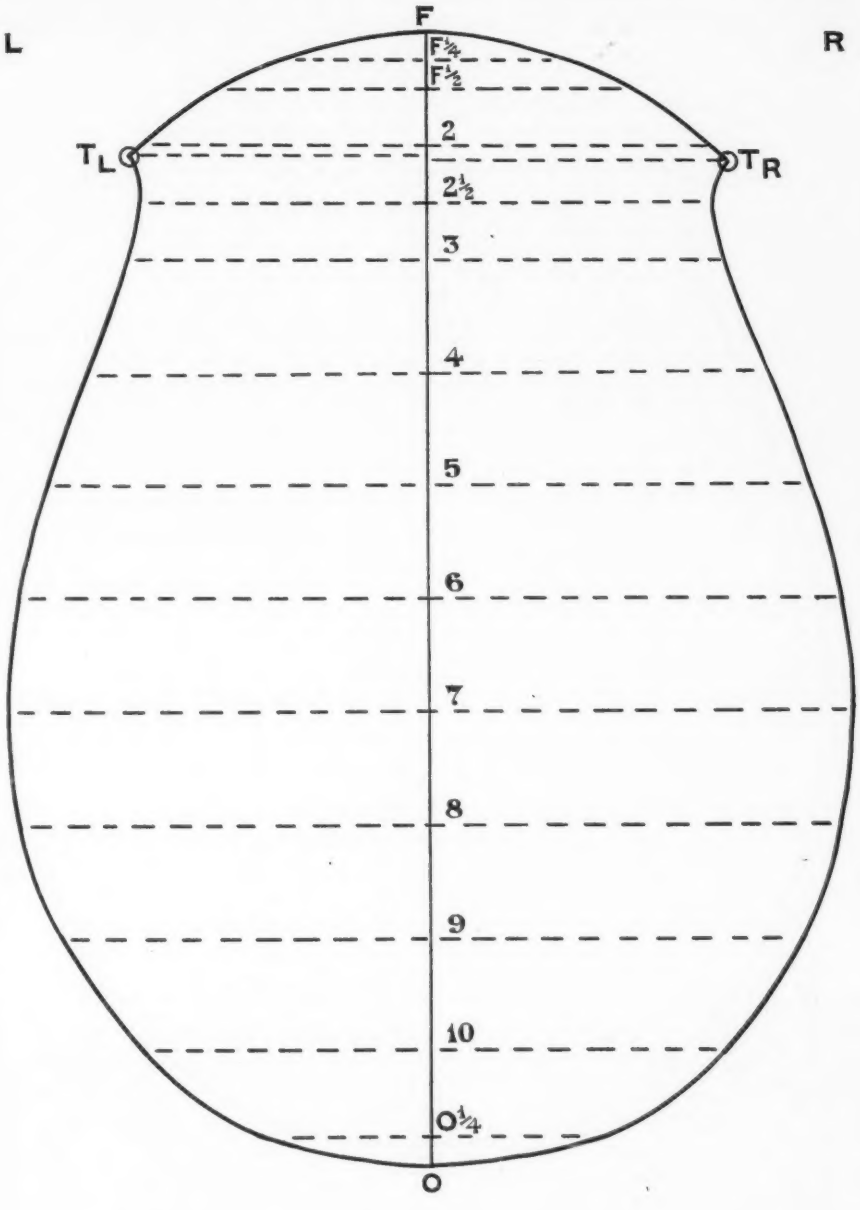


Fig. III. Horizontal Type Contour of 38 ♂ Egyptian Skulls from Sedment.

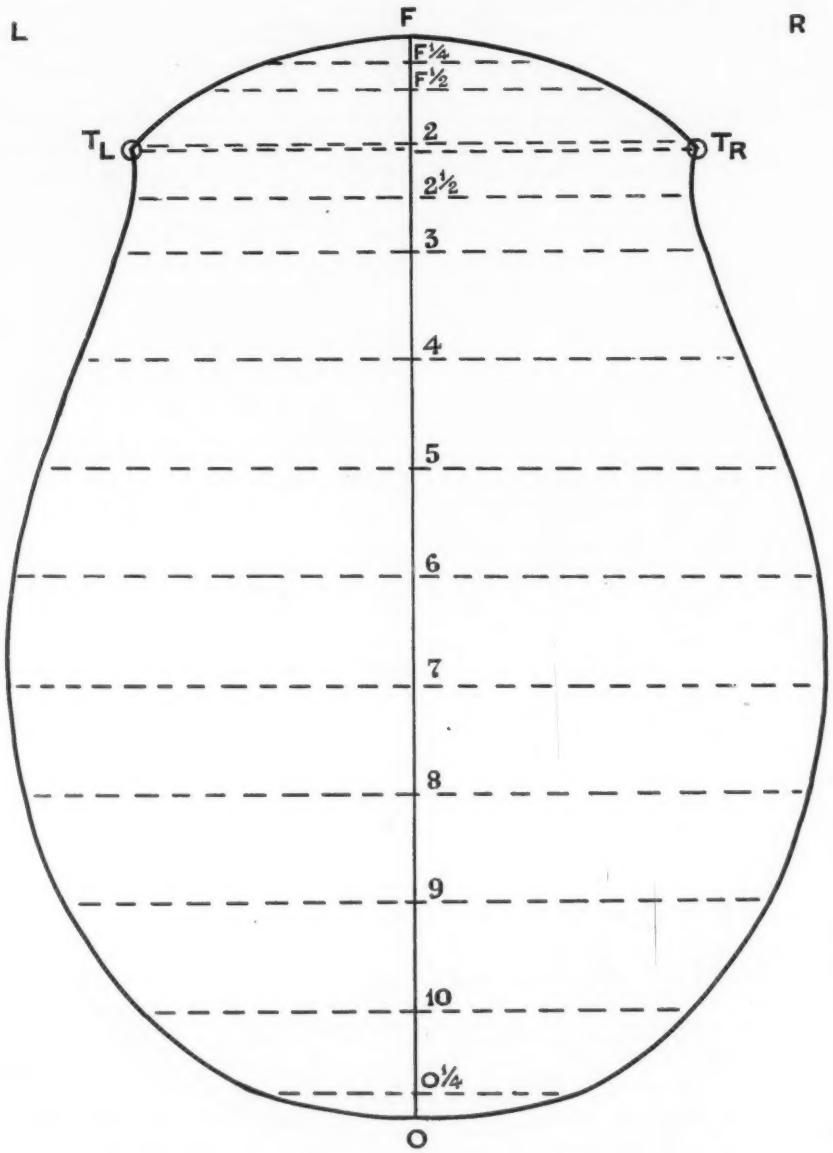


Fig. IV. Horizontal Type Contour of 30 ♀ Egyptian Skulls from Sedment.

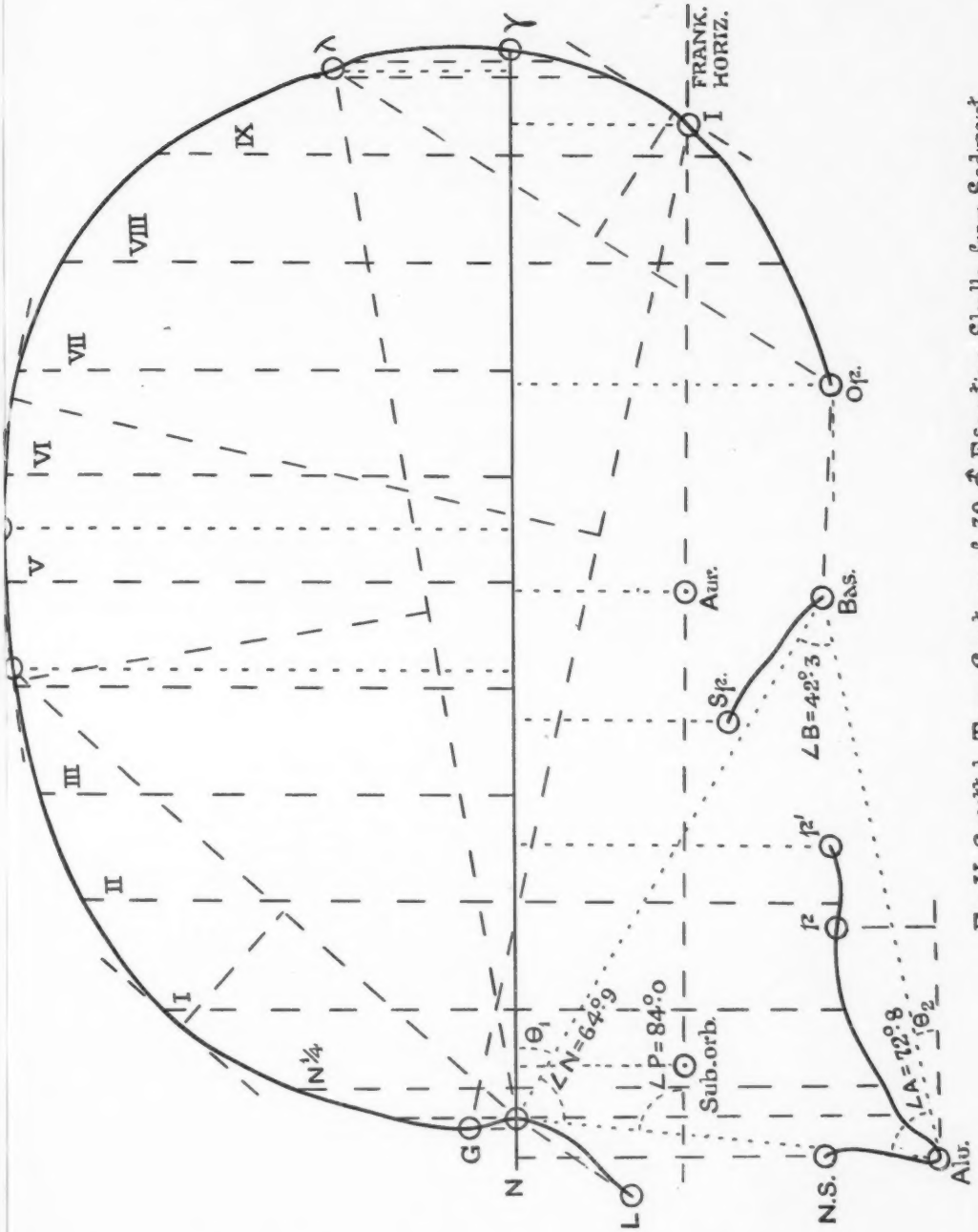
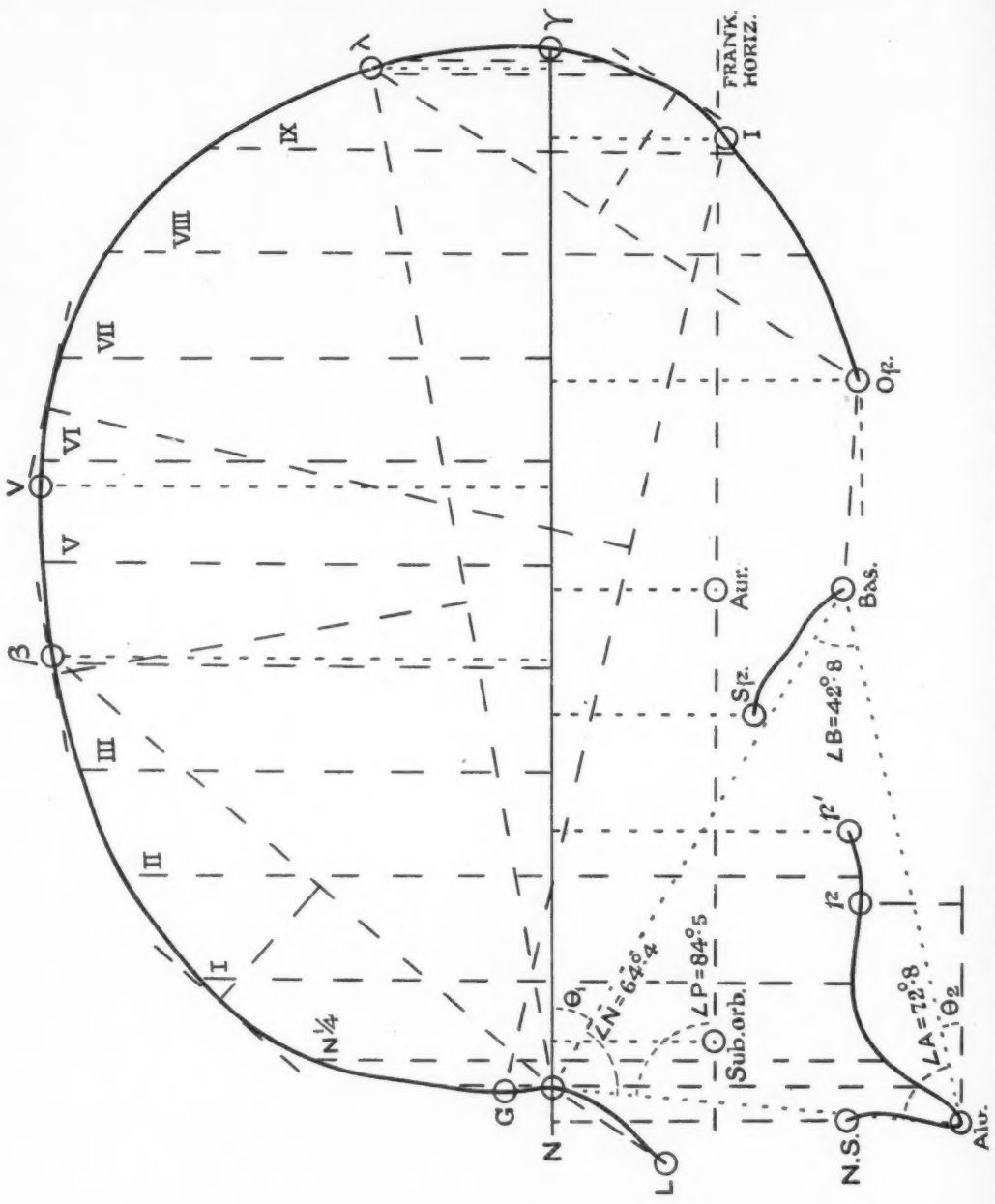


FIG. V. Sagittal Type Contour of 39 ♂ Egyptian Skulls from Sedment.


FRANK.
HORIZ.
Or.
Aur.
Bas.
Spr.
∠B=42°.8
∠P=84°.5
∠N=64°.4
∠A=72°.8
Sub.orb.
N.S.
Alv.
G
N
L
N¼
I
II
III
V
VI
VII
VIII
IX
β

by modern races is small and the Sedment values fall within the ranges given by the other types in all except one case. The distance of the parallel $F\frac{1}{2}$ from F—i.e. $\frac{1}{20}$th of FO—expressed as a percentage of the total length of that parallel, gives a measure of the curvature of the most anterior part of the frontal section. The highest ♂ index found is 13·6 for the Sedment type contour, but there are several others greater than 13. The lowest index, indicating the frontal bone which is most flattened transversely, is 11·0 (Prehistoric Chinese) and all the lowest values are for Asiatic races. By superposing the contours it is found that the Sedment type corresponds more closely to the horizontal section given for the 26th—30th Dynasty skulls from Gizeh than to the Badari or 1st Dynasty Egyptian figures.

The median sagittal contours (Figs. V and VI) again exhibit few distinctive features. As the series is an Egyptian one, the most characteristic are the unusual height and the flattened occipital section. The vertices lie between the 5th and 6th parallels and the same has been found for all other types as yet constructed. A number of indicial and angular measurements have been devised to aid in the comparison of the most important features of this section*. The ♂ and ♀ Sedment values were determined for each of these, and in all cases they fall within the ranges furnished by the type contours which had been constructed previously. Considerable differences are found from the sagittal figures available for other Egyptian series. The Sedment sections have quite the least protruding and flattest occiputs: they are also the greatest in height. A curious relationship is found on superposing the Sedment and Badari ♂ figures with the nasions and $N\gamma$ lines coincident. The outlines from nasion to lambda almost cover one another and no difference can be detected between the heights of the vertices. Below the lambda the Badari occipital section protrudes beyond the Sedment, but the outlines cross between the inion and opisthion. Quite marked differences are found between the outlines of the basi-occipital and the palate, the Sedment bones being immediately below the Badari but appreciably further removed from the $N\gamma$ line. The difference between the basio-bregmatic, or vertical heights, of the two types is thus due solely to differences associated with the base of the skull. The most significant differences between the Egyptian types are found for this section. The comparison of type contours confirms the conclusion suggested by direct measurements that the 9th Dynasty series from Sedment is not closely related to any one of the other three Egyptian ones which have been described in this way. More adequate comparative material will be needed in order to determine its racial affinities more exactly.

6. *Measurements of Mandibles.* There are 62 of the skulls in the Sedment series with more or less complete mandibles, 36 being ♂, 25 ♀ and 1 juvenile. Notes on the condition of the teeth and any dental anomalies noted are given in Appendix I. Measurements were taken according to the technique described by Morant†. Individual values are given in Appendix II and means in Table X. There is far less comparative material than for the skull, and no standard deviations or correlations

* See *Annals of Eugenics*, Vol. II. (1927), pp. 365—368.

† "A First Study of the Tibetan Skull," *Biometrika*, Vol. XIV. (1923), pp. 193—260.

TABLE IX.

Mean Measurements of Sagittal Contours.

Sex	Ordinates above $N\gamma$												
	$0=N$	$N\frac{1}{4}$	1	2	3	4	5	6	7	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{2}$
♂	19·9 (39)	36·8 (39)	58·9 (39)	72·4 (39)	80·0 (39)	83·8 (39)	86·1 (39)	86·2 (39)	83·2 (39)	75·4 (39)	59·1 (39)	33·6 (39)	26·8 (39)
♀	18·5 (30)	39·5 (30)	58·8 (30)	71·0 (30)	77·8 (30)	81·7 (30)	83·8 (30)	84·1 (30)	81·2 (30)	73·3 (30)	56·7 (30)	32·5 (30)	25·8 (30)

Sex	Ordinates below $N\gamma$								$N\gamma$	Vertex		β		Glabella		Occipital Point		λ		Sub-Orbital Point	
	$0=N$	$N\frac{1}{4}$	1	2	8	9	$\gamma\frac{1}{4}$	$\gamma\frac{1}{2}$		x from N	y	x from N	y	x from N	y	x from γ	y*	x from γ	y	x from N	y
♂	64·2 (38)	62·2 (38)	55·1 (37)	54·8 (36)	45·8 (37)	35·1 (39)	16·5 (39)	10·2 (39)	178·8 (39)	98·2 (39)	87·0 (39)	74·5 (39)	85·0 (39)	1·8 (39)	7·7 (39)	1·1 (39)	9·0 (39)	3·8 (39)	29·2 (39)	8·8 (39)	28·1 (39)
♀	60·3 (28)	54·6 (28)	49·2 (24)	50·8 (21)	42·5 (29)	30·4 (30)	14·6 (30)	9·0 (30)	171·5 (30)	98·8 (30)	84·6 (30)	70·0 (30)	82·0 (30)	0·9 (30)	8·2 (30)	0·7 (30)	7·4 (30)	3·9 (30)	29·6 (30)	7·3 (30)	26·4 (30)

Sex	Auricular Point		Opisthion		Inion		Basion		Alveolar Point		Nose							Palate			
	x from N	y	x from γ	y	x from γ	y	from N	from γ	from N	from Basion	NL	I	II	III	$LN\gamma\angle$	Max. Sub. x from N	Max. Sub. y	P' x from N	P' y	P x from Alv.	P y
♂	87·9 (39)	28·7 (39)	56·0 (36)	53·6 (36)	12·3 (39)	30·1 (39)	101·3 (37)	105·5 (37)	71·5 (38)	96·2 (37)	23·2 (25)	1·6 (37)	3·7 (35)	7·9 (27)	123°·7 (25)	2·8 (25)	8·7 (25)	45·8 (34)	53·0 (34)	38·4 (37)	16·7 (37)
♀	81·9 (30)	27·0 (30)	55·5 (27)	50·7 (27)	15·1 (30)	28·7 (30)	95·0 (30)	101·7 (30)	67·4 (29)	89·9 (29)	22·1 (14)	1·7 (29)	3·6 (28)	7·2 (21)	123°·6 (14)	2·3 (14)	7·8 (14)	42·1 (22)	48·8 (22)	35·6 (24)	16·7 (24)

Sex	Frontal Max. Sub. to $N\beta$		Occipital Max. Sub. to λOp		Max. Sub. to $N\lambda$		Max. Sub. to GI		Sp.		Mid. Sub. from Bas.-Sp. Chord		$N.S.$		Max. Sub. of Alv.-$N.S.$ Arc	
	x from N	y	x from λ	y	x from N	y	x from G	y	x from N	y	x from Bas.	y	x from N	y	x from N	y
♂	51·8 (39)	25·9 (39)	51·2 (36)	24·9 (36)	85·6 (39)	72·0 (39)	101·7 (39)	102·1 (39)	66·6 (39)	35·8 (39)	10·4 (37)	0·9 (37)	6·7 (38)	51·8 (38)	5·3 (38)	54·3 (38)
♀	50·5 (30)	25·5 (30)	44·3 (27)	24·1 (27)	80·9 (30)	69·0 (30)	92·1 (30)	99·1 (30)	61·3 (29)	33·1 (29)	10·3 (29)	1·4 (29)	5·7 (28)	48·3 (28)	4·4 (28)	52·2 (28)

* The occipital points are above the $N\gamma$ lines.

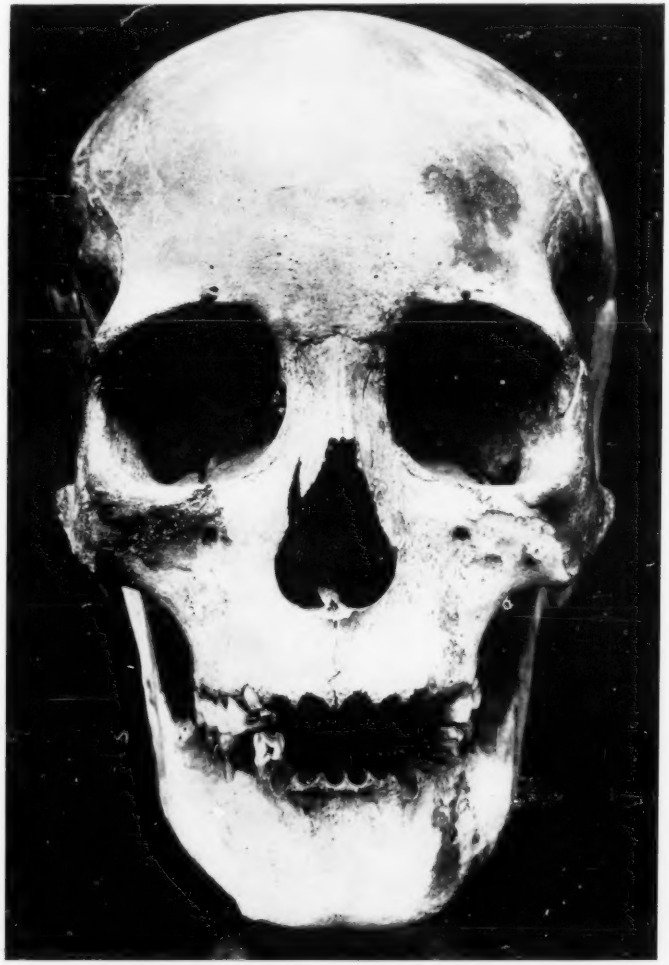
A

B

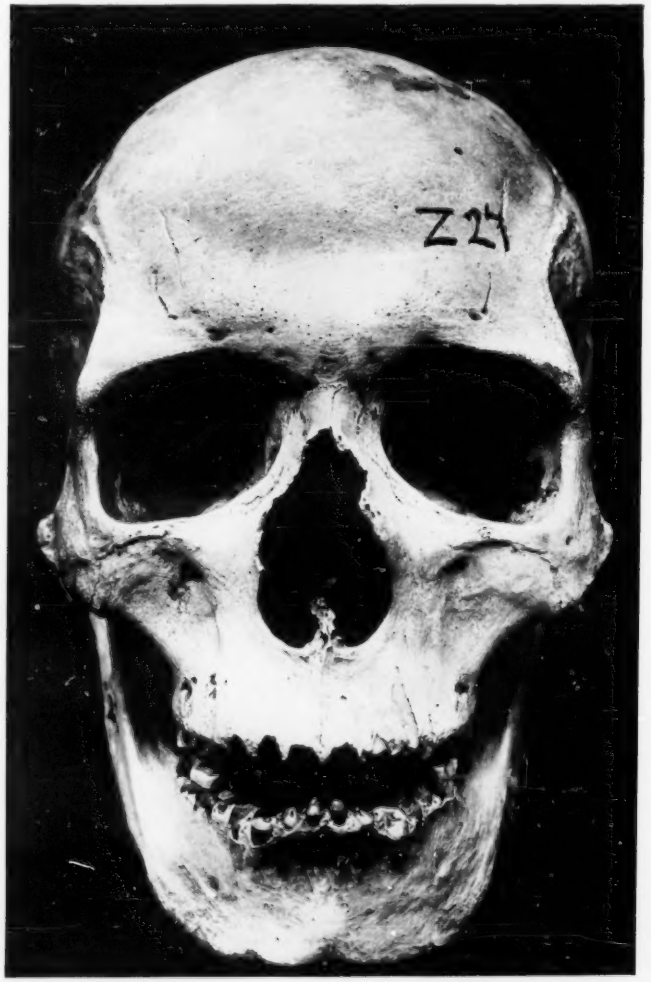
*	(30)	(30)	(27)	(27)	(30)	(30)	(30)	(30)	(29)	(29)	(29)	(29)	(28)	(28)	(28)	(28)

* The occipital points are above the $N\gamma$ lines.

A



B



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Normal Sedment Egyptian Skulls. *Norma facialis* (circa 0·8 natural size). A No. 21 ♀. B No. 27 ♂.

A

B

Normal Sedment Egyptian Skulls. *Norma lateralis.* A No. 21 ♀. B No. 27 ♂.

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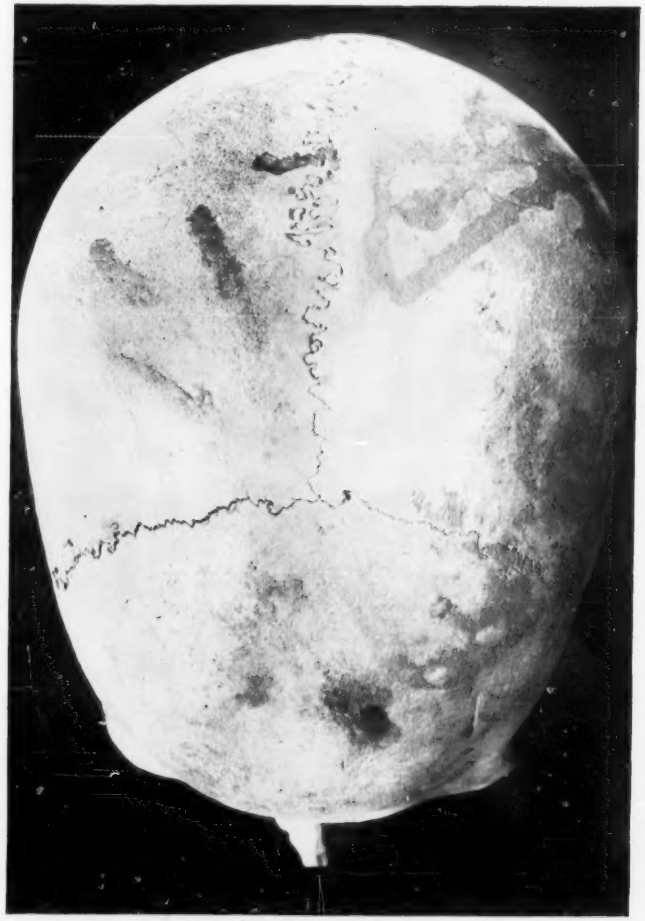
A
B

A



B

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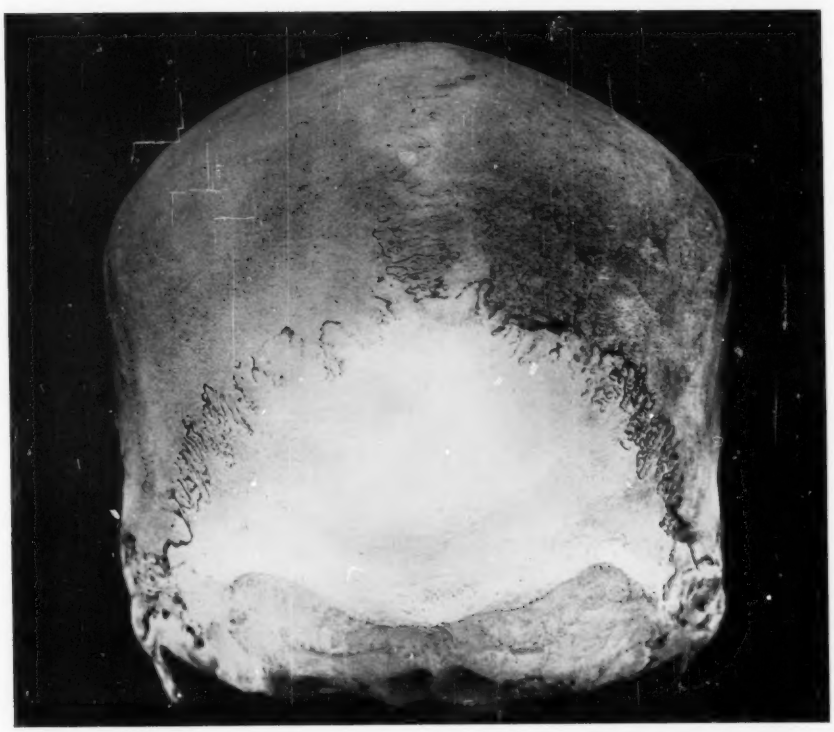
Normal Sedment Egyptian Skulls. *Norma verticalis* (circa 0·7 natural size). A No. 21 ♀. B No. 27 ♂.

A

B



Normal Sedment Egyptian Skulls. *Norma occipitalis* (circa 0·75 natural size).
A No. 21 ♀. B No. 27 ♂.

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A. Basi-occipital incisure on the left side. No. 65 ♂ (circa 1·5 natural size).

B. Large protruding ossicle above parieto-mastoid suture. No. 63 ♂? (circa natural size).

Anomalous Sedment Egyptian Skulls.

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Sex

♂
♀

Sex

♂
♀

Sex

♂
♀

Sex

♂
♀

of mandibular measurements are as yet available. It is not to be doubted, however, that measurements of the lower jaw will be extremely useful ultimately in aiding the discrimination of racial types and in other ways. Data determined by following

TABLE X.

Mean Measurements of Sedment Mandibles.

Sex		Lengths with Callipers														
		w_1	w_2	h_1	zz	$c_r c_r$	rb'	G_2'	$c_y c_r$	$g_o g_o$	$g_n g_o, l$	$g_n g_o, r$	$c_y l$	$c_y b$	$m_2 p_1$
♂	Mean	114·3	92·0	33·7	44·0	92·1	33·3	44·6	34·7	88·1	84·1	83·8	20·8	8·9	27·7
	No.	26	32	33	34	22	35	34	32	29	31	33	30	34	32
♀	Mean	107·8	85·6	30·9	42·5	84·9	30·4	43·2	31·6	81·1	78·4	78·9	19·1	8·5	27·0
	No.	18	21	22	22	17	24	21	23	19	19	20	21	23	20

Sex		Lengths with Callipers				Tape	Lengths and Heights on Mandible Board									
		$p_a d_t$	$p_a g_n$	$p_a d_f$	$g_n d_f$	$g_o p_a g_o$	ih	ih'	$c_r h$	$c_y h$	$d_t h$	$m_2 h$	$p_1 h$	$c_p l$	rl	ml
♂	Mean	28·3	8·2	25·3	30·7	191·6	48·3	13·3	67·7	57·6	36·9	26·9	31·5	75·8	63·0	102·6
	No.	32	36	32	32	29	35	32	33	35	32	34	33	36	35	36
♀	Mean	26·6	7·0	23·0	27·4	178·3	42·7	12·2	60·5	51·2	34·2	24·1	28·9	71·6	56·9	98·1
	No.	22	23	22	22	18	24	22	24	22	22	23	21	21	23	21

Sex		Indices								
		$100 c_r h/ml$	$100 c_r c_r/ml$	$100 g_o g_o/c_p l$	$100 r'b/rl$	$100 c_y b/c_y l$	$100 g_o g_o/c_r c_r$	$100 c_y h/c_r h$	$100 i'h/c_y c_r$	$100 d_t h/c_r h$
♂	Mean	66·5	90·8	116·5	53·0	43·7	96·2	85·1	38·5	53·8
	No.	33	22	29	34	30	21	32	32	29
♀	Mean	61·8	86·5	114·2	53·9	44·2	95·4	84·5	38·8	57·1
	No.	21	16	18	22	21	17	22	22	21

Sex		Angles								
		$M\angle$	$R\angle$	$G\angle$	$C\angle$	$C'\angle$	$L\angle$	$L'\angle$	$F\angle$	$S\angle$
♂	Mean	121°·0	77°·1	63°·2	73°·9	75°·5	94°·1	93°·1	90°·2	17°·6
	No.	36	34	28	31	32	25	25	25	24
♀	Mean	123°·4	74°·6	62°·6	70°·4	71°·3	94°·0	92°·3	90°·5	23°·9
	No.	23	23	18	22	22	17	17	17	17

precisely the same technique as we have used have only been published for one other Egyptian series: that is the Badari studied by Stoessiger*. Mean values of the principal measurements for the Egyptian and for a Tibetan and an Anglo-Saxon series are given in Table XI. Some of the differences between the extremes are

TABLE XI.

Mean Male Mandibular Measurements for Egyptian and other Series.

	9th Dynasty Egyptians (Sedment)	Predynastic Egyptians (Badari)	Tibetan A†	Anglo-Saxons‡
Maximum breadth at condyles (w_1)	114·3 (26)	109·5 (30)	117·0 (25)	123·7 (25)
" " angles (w_2)	92·0 (32)	88·8 (32)	96·2 (25)	103·2 (45)
Height of symphysis ... (h_1)	33·7 (33)	32·6 (34)	30·6 (25)	33·1 (40)
Minimum breadth of ramus (rb')	33·3 (35)	33·6 (39)	32·1 (25)	33·2 (61)
Condylion to coronion ... ($c_y c_r$)	34·7 (32)	33·8 (37)	34·2 (25)	33·9 (40)
Maximum length of condyle ($c_y l$)	20·8 (30)	20·3 (36)	18·8 (25)	21·7 (38)
Height of coronion ... ($c_r h$)	67·7 (33)	61·8 (33)	60·8 (25)	65·7 (48)
Height of incisura ... (ih')	13·3 (32)	12·2 (33)	15·0 (25)	13·6 (35)
Length of ramus ... ... (rl)	63·0 (35)	57·6 (33)	58·4 (25)	64·0 (45)
Total projective length ... (ml)	102·6 (36)	101·2 (33)	105·2 (25)	107·2 (31)
$100 c_r h/ml$	66·5 (33)	61·0 (32)	57·8 (25)	60·9 (27)
$100 rb'/rl$	53·0 (34)	58·5 (33)	55·3 (25)	51·5 (45)
$M\angle$	121°·0 (36)	120°·0 (34)	125°·3 (25)	120°·3 (47)

surprisingly large. For the condylar breadth (w_1) the greatest mean exceeds the least by 14·2 mm. and for the angular breadth (w_2) the extreme difference is 14·4 mm. These values are almost as great as the *maximum* inter-racial difference found for the larger diameters of the skull such as the length, breadth and height. Some of the other characters—notably rb' and $c_y c_r$—are almost constant for the four series in the table. It is probable that several of the differences between the Sedment and Badari means are significant.

7. *Conclusions.* The series of 9th Dynasty skulls from Sedment, in Upper Egypt, is not more variable than other Egyptian dynastic ones. Judging by a generalised measure of resemblance—Professor Karl Pearson's coefficient of racial likeness—the type is more closely related to that of 4th and 5th Dynasty skulls from neighbouring graveyards at Deshasheh and Medum than to any other which has been adequately described. In spite of this close link, the Sedment series stands still closer to one of modern Cretans than to any Egyptian series at present available. No other close connections have been found with European races, and all those of a rather less intimate order are with dynastic series from Upper Egypt except one with a modern series from Cairo. The last, and the modern Cretans, are only connected

* "A Study of the Badari Crania recently excavated by the British School of Archaeology in Egypt." *Biometrika*, Vol. XIX. (1927), pp. 110—150.

† *Ibid.* Vol. XVI. (1924), pp. 103 and 104.

‡ *Ibid.* Vol. XVIII. (1926), p. 96.

Calvaria

Serial No.	Grave No.	Sex	*F*	*L*	*B*	*B′*	*H*	*H′*	*OH*	*LB*	*Q′*	*S*	S_1	S_2	S_3
2	No Mark	♂?	176·5	176·0	135·0	95·5	—	—	—	—	—	—	125·5	125·0	—
3	No Mark XI (sic)	♂	184·5	185·5	138·0	96·0	138·5	138·0	112·6	104·5	308·5	375·0	122·5	138·5	114·0
7	391 IX	♂	185·5	190·0	140·5	103·3	142·0	142·0	121·5	109·5	330·5	383·5	131·5	127·5	124·5
8	296 IX	♂	187·5	188·0	142·0	91·8	136·5	134·0	117·8	101·0	324·0	383·0	129·0	132·0	122·0
9	1591 IX	♂	168·0	167·0	136·0	94·0	136·0	135·0	110·3	98·0	303·5	347·0	118·0	121·0	108·0
10	1613 IX	♂?	182·5	187·0	143·5	94·6	135·5	134·0	114·5	98·5	321·0	374·5	130·5	131·0	113·0
11	1550 IX	♂	185·5	189·5	142·0	96·3	139·0	138·0	116·0	—	324·0	—	—	131·0	116·0
13	1509 XI (sic)	♂?	176·5	179·5	135·0	89·6	142·0	140·5	118·5	100·0	321·5	—	135·0?	134·0	—
14	1804 IX	♂	183·5	186·0	138·5	99·1	150·5	149·5	120·4	107·0	328·0	386·0	132·5	132·5	121·0
16	389 IX	♂	182·5	184·0	141·0	90·6	141·0	141·0	115·7	106·0	312·5	381·0	134·5	122·0	124·5
17	1618 IX	♂?	173·0	177·5	140·5	87·0	141·5	141·5	115·5	96·0	307·0	371·0	132·5	127·0	111·5
19	1540 IX	♂?	179·5	178·0	140·0	92·1	138·0	137·0	114·3	93·5	314·5	376·5	128·0	127·0	121·5
22	388 IX	♂?	177·5	179·5	138·0	89·6	137·0	137·5	114·5	103·5	311·0	371·5	133·0	134·5	104·0
23	536 IX	♂	173·5	177·0	138·0	90·4	144·0	143·5	116·5	104·5	316·0	371·5	129·5	129·0	113·0
26	289 IX	♂?	169·5	172·5	139·5	91·0	136·0	135·0	112·4	101·0	312·5	349·5	122·0	118·5	109·0
27	1625 IX	♂	184·0	188·0	138·0	93·8	138·0	138·0	118·3	104·5	319·0	382·5	134·0	130·0	118·5
28	293 IX	♂	185·5	189·0	132·0	96·1	138·0	138·0	120·5	101·5	319·0	383·0	139·5	135·0	108·5
29	1603 IX	♂	181·5	186·0	133·0	98·6	141·0	139·5	118·6	107·5	315·0	372·0	120·5	135·0	116·5
31	1614 IX	♂	174·5	178·0	140·5	86·6	138·0	136·5	108·7	103·5	295·5	360·5	126·5	112·0	122·0
32	1653 IX	♂	178·5	180·5	134·0	100·5	142·0	141·0	114·5	105·0	313·5	361·5	127·0	128·0	106·5
35	1704 IX	♂?	177·5	178·0	133·0	88·9	—	—	111·7	—	310·0	—	130·0	132·0	—
38	1534 IX	♂	178·5	179·0	137·5	97·4	131·5	131·5	109·0	102·0	303·0	363·0	126·0	117·0	120·0
39	1601*a* IX	♂	189·5	195·0	144·0	100·8	145·0	144·5	121·0	109·0	330·0	398·0	137·0	130·0	131·0
41	1542 IX	♂?	174·0	173·5	139·5	83·2	136·0	136·0	113·1	95·0	316·0	366·0	139·0	121·0	106·0
42	1549 IX	♂	178·5	180·0	140·5	95·4	130·0	129·0	114·3	97·5	320·0	368·0	134·0	126·0	108·0
44	1802 IX	♂	181·5	183·0	138·0	91·2	138·0	137·0	117·0	96·5	314·5	377·5	127·5	133·0	117·0
45	1600? IX	♂?	178·5	180·0	134·5	82·3	134·5	133·5	112·9	101·5	303·0	369·5	129·5	132·0	108·0
46	1555 IX	♂	178·0	184·0	131·0	90·4	136·0	133·5	110·3	104·0	298·0	362·5	116·0	131·5	115·0
48	1529 IX	♂	183·0	183·5	135·5	89·8	133·0	133·0	108·5	102·5	302·5	369·0	119·0	139·0	111·0
50	1562 IX	♂	169·5	174·5	132·0	87·7	134·5	134·0	113·6	93·5	309·0	364·5	129·5	136·0	99·0
52	1727*a*? IX	♂	184·5	184·0	143·0	101·7	142·0	141·5	122·5	97·5	327·5	384·5	132·5	138·0	114·0
56	538 IX	♂	173·0	178·0	137·0	90·0	138·0	137·0	115·2	97·0	318·0	365·0	126·0	139·0	100·0
57	1718 IX	♂?	177·5	179·5	145·5	92·2	137·5	137·0	114·3	101·0	313·0	369·5	128·0	122·5	119·0
63	1527 IX	♂?	178·5	177·0	135·0	89·0	132·5	132·0	110·7	94·0	297·5	375·0	131·0	129·0	115·0
64	1587 IX	♂?	173·0	175·0	130·0	88·6	130·0	128·0	111·0	97·0	304·5	358·0	117·0	135·0	106·0
65	1514 IX	♂	185·0	194·0	139·0	91·4	135·0	133·5	115·2	97·0	320·0	387·0	131·0	128·5	127·5
66	1593 IX	♂	175·5	178·0	141·0	95·6	146·0	144·0	119·5	103·0	320·0	372·0	128·0	120·0	124·0
68	1579 IX	♂	184·0	185·0	141·0	88·6	139·5	140·0	119·8	97·0	322·0	391·5	139·0	140·0	112·5
70	1700 IX	♂	183·5	188·5	147·0	89·8	141·0	139·0	117·8	100·0	327·0	393·5	131·5	141·0	121·0
71	1605 IX	♂?	190·5	189·5	142·0	93·7	137·0	136·0	119·2	100·0	328·5	391·5	128·0	136·5	127·0
Means			179·6	181·9	138·3	92·6	138·2	137·4	115·3	100·8	314·9	373·8	129·0	129·9	115·0
No. of cases			40	40	40	40	38	38	39	37	39	36	39	40	37
1	1632 IX	♀	167·5	168·0	127·0	86·2	126·0	125·0	103·4	93·0	285·0	338·5	117·5	108·0	113·0
4	No Mark IX	♀?	179·5	179·0	139·0	88·7	136·0	134·0	115·8	96·0	316·0	378·0	127·0	133·0	118·0
5	15?? IX	♀	175·0	173·5	129·5	82·4	134·0	132·0	108·5	98·0	297·0	361·0	118·0	118·0	115·0
6	1586 IX	♀?	173·5	173·0	130·0	84·5	134·0	132·0	112·5	98·0	299·0	355·5	118·0	131·0	106·5
12	1526 IX	♀?	167·5	167·5	122·5	81·1	120·5	120·0	107·2	90·0	292·5	342·5	123·0	122·0	97·0
15	1569 IX	♀?	183·5	184·0	146·0	89·9	135·5	134·0	114·5	104·0	316·0	—	129·0	133·0	—
18	1715 IX	♀?	167·0	165·5	128·0	82·8	133·0	132·0	110·0	93·5	296·5	348·0	117·0	115·0	116·0
21	1601*d* IX	♀?	179·0	178·5	133·0	84·9	130·5	129·5	109·3	96·5	302·0	368·0	121·0	129·0	117·5
24	1540″ IX	♀	164·0	164·0	129·5	85·7	128·0	125·5	106·5	89·0	290·0	343·5	113·0	124·5	106·0
25	1506 IX	♀?	173·5	173·0	132·0	85·0	138·0	136·0	111·7	97·0	303·0	359·0	118·0	128·0	113·0
30	1541 IX	♀	167·5	168·5	125·5	86·3	123·5	123·0	103·8	95·5	285·0	346·0	116·0	124·5	105·5
33	1671 IX	♀?	171·0	172·0	134·0	85·6	134·0	133·0	111·6	92·5	306·0	359·0	122·5	117·5	119·0
34	1537 IX	♀?	185·0	184·5	145·0	96·8	130·5	130·0	115·7	90·5?	322·0	—	135·0	132·0	—
36	1558 IX	♀?	170·0	170·5	138·5	85·0	135·5	134·5	113·4	94·0	312·5	367·0	130·0	124·5	112·0
37	400 IX	♀?	173·5	172·0	144·0	95·0	132·5	132·0	114·5	85·5	316·0	371·5	128·0	125·5	118·0
40	1635	♀	163·5	162·0	131·0	83·2	127·5	127·0	100·5	97·0	277·0	328·0	110·0	113·0	105·0
43	387 IX	♀?	172·0	173·0	130·5	90·0	131·0	131·0	114·3	89·0	312·5	365·0	129·0	125·0	111·0
47	1504 IX	♀?	173·5	175·0	130·0	80·5	135·0	133·0	113·4	99·5	306·0	359·5	118·0	126·5	115·0
49	1803 IX	♀?	177·5	176·0?	133·5	96·0	130·0	129·0	109·5	92·0	295·0	373·0	134·0	129·0	110·0
51	1521 IX	♀?	173·0	173·0	130·0	88·0	137·0	137·0	115·2	99·5	307·0	363·0	124·0	131·0	108·0
53	1541 IX	♀?	175·5	178·0	137·0	91·9	145·0	144·0	121·6	95·0	325·0	—	141·0	127·5	—
54	1540‴ IX	♀?	177·0	179·0	138·0	90·8	137·5	136·5	111·5	99·0	308·5	366·5	127·0	128·0	111·5
55	+60*z* IX	♀	169·5	171·0	127·0	83·7	129·5	129·0	109·0	96·0	296·5	351·0	119·0	122·0	110·0
58	1538 IX	♀	169·5	169·5	128·0	88·7	123·0	121·5	104·8	92·5	293·0	344·0	119·0	122·0	103·0
59	1601*e* IX	♀	176·5	174·0	134·0	83·2	133·0	131·0	109·5	96·0	298·5	365·0	119·0	139·0	107·0
60	1731 IX	♀	173·0	173·0	132·0	88·6	129·5	128·0	107·6	91·0	294·0	359·0	124·0	117·0	118·0
61	1588 IX	♀?	172·5	172·0	134·5	89·0	130·5	130·0	108·3	95·5	300·0	356·5	126·0	112·0	118·5
62	1672 IX	♀	172·5	172·0	134·0	89·0	125·0	124·0	107·5	92·0	294·0	352·5	122·5	120·0	110·0
67	1581 IX	♀?	171·0	168·5	132·0	85·0	131·0	128·0	108·5	96·0	288·5	349·5	115·0	128·5	106·0
69	1522 IX	♀?	179·0	178·5	140·5	93·2	134·5	133·5	112·5	98·0	310·5	370·5	126·5	134·0	110·0
20	1601*b* IX	Juvenile sex?	181·5	180·0	140·5	87·6	132·5	131·0	113·3	92·0	308·0	375·5	125·5	132·0	118·0
Means*			173·1	172·9	133·5	87·4	131·7	130·5	110·4	94·7	301·5	357·1	123·2	124·7	111·1
No. of cases			30	30	30	30	30	30	30	30	30	27	30	30	27

* No. 20 excluded.

							Face				Nose									
S_1	S_2	S_3	S_1'	S_2'	S_3'	U	PH	$G'H$	GB	J	NH'	$\frac{NH}{R}$	$\frac{NH}{L}$	NB	DS	DC	DA	SS	SC	O_1R	O_1L
125·5	125·0	—	108·5	111·0	—	505·0	18·6	68·0	97·1	125·0	49·2	49·8	49·9	24·5	13·7	25·2	39·2	3·7	14·1	41·7	42·2
122·5	138·5	114·0	109·1	122·5	98·8	518·0	19·5	72·8	91·6	128·5	54·9	52·7	52·8	24·2	11·7	20·4	32·6	3·5	8·1	42·8	42·8
131·5	127·5	124·5	117·2	114·7	107·4	534·0	23·2	77·0	103·0	135·0	54·1	54·7	54·6	24·6	12·3	23·8	36·5	5·1	10·3	45·8	46·4
129·0	132·0	122·0	111·8	117·0	97·2	526·0	20·2	73·8	97·7	—	53·6	53·0	54·0	24·6	10·1	19·7	29·4	2·5	9·5	41·2	42·8
118·0	121·0	108·0	103·1	105·5	95·7	486·0	22·0	72·6	89·1	130·5	53·0	50·8	50·5	20·7	11·7	20·8	32·1	3·4	9·3	37·8	40·2
130·5	131·0	113·0	116·4	118·7	96·6	525·0	17·6	72·1	88·0	—	54·3	54·2	54·6	22·6	12·5	18·1	23·8	5·3	11·4	43·2	41·0
—	131·0	116·0	—	117·3	101·5	535·0	—	—	88·2	—	—	—	—	25·2	—	—	—	—	—	45·0	—
135·0?	134·0	—	116·4	116·7	—	501·0	23·8	71·6	92·4	—	47·4	50·5	50·4	24·2	—	—	—	—	—	41·0	—
132·5	132·5	121·0	117·0	117·7	102·6	523·0	22·0	73·3	95·8	135·0	51·3	51·2	53·0	25·3	—	—	—	4·4	10·3	46·0	44·7
134·5	122·0	124·5	120·2	109·5	105·4	515·0	18·4	68·2	93·8	127·5	50·4	50·6	50·5	22·0	11·7	22·6	32·6	3·6	10·8	42·0	40·6
132·5	127·0	111·5	118·6	112·4	101·0	505·0	22·0	73·7	91·8	120·0	52·2	51·6	51·7	24·3	9·8	20·9	28·4	2·3	9·1	40·2	39·8
128·0	127·0	121·5	109·0	114·2	103·4	506·0	23·4	72·0	91·2	118·5	48·5	49·6	49·2	23·8	11·9	20·9	33·6	1·7	7·0	42·0	40·2
133·0	134·5	104·0	113·0	117·0	91·4	505·0	20·1	75·0	87·6	123·0	52·6	55·4	56·0	21·0	11·2	21·6	32·2	3·7	11·0	40·2	39·6
129·5	129·0	113·0	116·0	112·1	98·4	503·0	25·6	79·0	101·4	129·5	53·2	53·0	52·8	23·2	12·9	19·6	34·4	2·0	7·0	40·3	40·4
122·0	118·5	109·0	109·6	107·6	91·3	496·0	17·5	68·8	90·6	124·5?	51·3	53·6	53·9	24·9	10·6	17·7	33·5	4·2	7·4	43·0	40·2
134·0	130·0	118·5	118·0	116·9	99·2	520·0	23·2	76·0	97·8	130·0	52·0	53·8	53·7	26·8	13·0	21·3	36·5	4·6	10·2	43·2	43·2
139·5	135·0	108·5	122·5	120·0	92·4	517·0	19·6	71·0	98·7	129·0	51·2	54·4	54·2	25·2	13·6	23·2	37·0	5·3	12·7	41·8	43·0
120·5	135·0	116·5	107·9	122·8	96·4	515·0	19·2	70·0	97·2	131·0	50·7	53·0	52·7	26·3	10·0	18·7	29·9	5·8	9·4	47·8	47·2
126·5	112·0	122·0	113·0	99·6	103·0	498·0	18·8	70·0	93·4	124·0	51·2	52·4	52·4	24·5	11·6	24·3	34·8	4·2	10·6	42·8	42·8
127·0	128·0	106·5	113·6	114·6	95·5	509·0	19·7	76·4	101·6	130·0	57·4	56·2	55·6	25·0	13·1	20·7	33·1	4·1	7·4	44·7	45·8
130·0	132·0	—	112·0	116·8	—	500·0	16·9	66·7?	—	—	49·4	48·5	48·7	24·6	—	—	—	4·5	11·4	—	38·8
126·0	117·0	120·0	111·6	105·5	96·5	505·0	22·8	75·0	94·9	126·0	49·7	52·9	54·3	28·3	12·6	21·2	32·0	3·5	9·5	43·8	42·2
137·0	130·0	131·0	123·0	118·4	107·2	537·0	24·5	81·0	102·0	138·0	56·0	56·7	56·8	27·8	15·0	26·9	40·0	5·6	13·0	46·2	46·0
139·0	121·0	106·0	113·8	108·8	93·8	501·0	18·8	62·0	86·5	121·0	42·3	43·0	43·3	21·3	10·0	19·0	28·0	1·7	6·7	38·6	38·4
134·0	126·0	108·0	115·8	112·6	92·8	522·0	17·2	73·8	96·3	127·5	54·6	54·4	55·5	26·8	12·5	22·4	32·9	4·4	9·1	41·6	41·6
127·5	133·0	117·0	113·2	116·5	100·0	518·0	20·3	75·3	93·8	128·5	53·7	53·2	51·6	23·7	11·3	19·3	31·7	4·7	7·9	40·8	40·0
129·5	132·0	108·0	113·8	114·6	94·0	501·0	15·6?	65·6	88·4	—	47·6	49·8	50·5	28·0	9·9	23·2	31·0	2·9	8·7	39·9	39·8
116·0	131·5	115·0	104·5	117·7	97·9	502·0	21·3	73·8	95·8	125·0	52·5	53·4	53·4	25·3	13·0	18·2	36·5	4·8	8·5	43·9	43·8
119·0	139·0	111·0	102·9	122·3	93·7	509·0	21·0	75·3	88·9	—	54·7	54·0	54·6	24·9	12·3	22·6	36·6	3·1	9·9	43·0	44·0
129·5	136·0	99·0	111·6	119·9	91·8	487·0	16·1	61·6	87·9	122·0?	45·7	46·2	46·6	24·6	12·3	18·6	34·0	4·6	9·9	40·0	41·0
132·5	138·0	114·0	115·4	124·3	93·7	521·0	15·9	68·6	93·3	127·5	51·3	52·0	52·0	24·6	11·1	21·2	33·0	5·3	11·0	43·2	42·0
126·0	139·0	100·0	113·6	118·5	91·0	503·0	18·4	67·8	91·4	126·0?	49·6	49·6	49·5	23·0	13·6	21·3	37·6	4·0	8·2	40·0	40·0
128·0	122·5	119·0	113·2	111·4	99·8	515·0	18·6	67·7	93·0	—	50·0	49·1	49·3	22·2	—	—	—	4·3	10·3	43·6	43·2
131·0	129·0	115·0	113·8	115·6	96·4	501·0	—	—	—	—	—	—	—	—	—	—	—	2·4	9·7	42·2	—
117·0	135·0	106·0	103·4	117·0	92·7	480·0	18·3	68·2	87·8	117·0	49·6	48·7	48·6	23·5	10·1	21·1	30·0	3·5	9·6	39·8	39·4
131·0	128·5	127·5	117·9	115·5	111·0	534·0	18·6	71·8	101·0	130·5	53·0	53·2	54·2	28·1	14·0	20·8	35·0	4·5	9·8	42·6	41·2
128·0	120·0	124·0	114·1	110·0	108·0	510·0	18·1	74·8	102·6	131·0	57·0	56·9	56·6	24·6	10·4?	22·3	32·0?	3·5	7·5	45·2	43·0
139·0	140·0	112·5	118·1	123·0	98·3	522·0	18·4	65·8	—	—	48·0	47·4	46·3	21·8	—	—	—	4·6	8·2	—	39·3
131·5	141·0	121·0	116·1	124·4	105·4	526·0	22	70·0	90·2	—	47·5	47·4	47·7	24·2?	—	—	—	—	—	41·3	—
128·0	136·5	127·0	111·6	121·3	105·2	533·0	17·6	69·7	93·6	129·0	52·4	52·8	52·7	24·8	9·8	20·4	30·6	2·7	8·0	41·1	41·6
129·0	129·9	115·0	113·3	115·6	98·6	511·7	19·9	71·5	93·9	127·2	51·4	51·8	52·0	24·5	11·9	21·2	33·1	3·89	9·5	42·4	41·9
39	40	37	39	40	37	40	38	38	37	29	38	38	38	39	32	32	32	37	37	38	36
117·5	108·0	113·0	103·8	98·8	95·8	474·5	16·7	66·9	84·1	110·0	50·0	50·6	50·6	25·9	—	—	—	3·4	9·1	40·4	40·6
127·0	133·0	118·0	111·0	117·7	100·8	506·0	17·6	67·5	88·5	112·5	48·8	48·7	48·6	22·8	11·6	18·2	29·6	3·0	9·1	41·9	41·6
118·0	118·0	115·0	108·9	105·6	99·2	487·0	16·2	65·4	87·1	114·0	49·6	49·4	49·5	22·5	10·7	18·8	28·0	2·9	8·8	38·6	38·0
118·0	131·0	106·5	103·0	116·0	92·4	484·0	18·5	69·1	91·2	119·0	52·0	49·6	49·7	21·5	—	—	—	4·2	8·4	41·2	40·6
123·0	122·0	97·0	103·8	109·2	83·6	466·0	12·8	55·1	81·0	106·0	41·3	41·4	41·3	22·8	7·9	17·5	23·8	2·0	7·6	35·8	35·6
129·0	133·0	—	114·7	116·4	—	523·0	19·5	73·0	94·5	125·0	53·2	53·4	53·8	22·8	11·3	20·6	31·0	4·5	11·6	43·2	43·0
117·0	115·0	116·0	102·8	103·6	100·8	472·0	20·0	66·7	87·8	119·0	46·2	46·8	47·4	25·6	12·1	17·6	32·6	5·1	11·0	40·0	38·2
121·0	129·0	117·5	106·6	115·5	97·8	496·0	18·0	71·8	94·5	120·5	54·0	53·4	53·2	22·8	12·3	19·8	33·4	4·8	9·7	41·0	40·7
113·0	124·5	106·0	106·0	110·0	93·0	465·0	16·9	61·4	88·7	113·5	43·1	45·3	45·8	26·2	—	—	—	4·6	10·4	39·6	39·8
118·0	128·0	113·0	103·5	114·8	98·2	486·0	19·6	64·4	92·5	—	44·3	44·8	45·2	24·6	10·7	21·4	26·4	3·4	10·1	40·2	39·6
116·0	124·5	105·5	103·2	108·3	89·7	476·0	16·3	63·4	87·7	118·0	47·5	47·0	47·9	23·6	10·1	19·3	30·0	5·0	10·5	37·8	36·0
122·5	117·5	119·0	110·0	107·9	101·6	487·0	18·7	64·8	91·1	117·0?	46·4	46·5	46·6	23·0	10·9	19·0	31·2	2·0	8·2	41·0	41·6
135·0	132·0	—	115·5	118·9	—	527·0	19·4	67·0?	90·0	—	47·8	48·7	49·5	24·8	—	—	—	—	—	41·6	41·2
130·0	124·5	112·0	113·3	109·6	95·0	496·0	21·1	71·8	93·2	119·0	51·2	50·3	50·2	23·0	11·3	17·5	29·2	3·1	8·0	39·4	39·0
128·0	125·5	118·0	110·2	112·5	98·4	508·0	17·0	66·6	89·4	121·0	49·8	49·7	49·5	23·5	10·3	20·1	30·0	4·3	12·5	41·3	41·0
110·0	113·0	105·0	96·6	102·5	91·4	463·0	19·2	65·6	88·2	111·0	46·8	47·0	46·4	21·0	11·1	16·4	28·4	4·5	8·4	38·2	37·8
129·0	125·0	111·0	113·6	108·9	97·8	503·0	18·6	66·4	85·6	118·0	48·4	48·4	47·5	22·2	12·2	20·0	32·7	2·8	9·7	41·7	43·0
118·0	126·5	115·0	103·8	110·4	99·1	487·0	19·0	65·2	88·7	118·0?	48·7	48·6	47·5	25·9	10·8	21·6	33·6	4·1	11·3	39·0	38·0
134·0	129·0	110·0	114·3	113·9	91·6	502·0	22·7	71·4	—	—	48·8	48·8	49·3	22·7	—	—	—	3·8	9·6	41·8	—
124·0	131·0	108·0	109·9	115·5	95·0	487·0	17·9	66·4	98·2	123·0	48·0	47·7	47·1	26·6	10·8	22·7	32·9	4·2	12·8	37·0	37·7
141·0	127·5	—	123·4	112·1	—	499·0	18·4?	70·2	88·4	124·0	49·5	48·8	48·5	25·1	12·7	23·1	37·4	4·3	10·2	41·2	40·0
127·0	128·0	111·5	110·8	112·9	98·6	509·0	20·6	70·0	95·4	124·5	48·2	50·7	49·5	23·6	—	—	—	3·2	9·1	43·0	42·0
119·0	122·0	110·0	103·5	108·1	95·0	480·0	—	—	—	—	—	—	—	—	11·4	19·5	28·2	4·1	10·0	—	35·9
119·0	122·0	103·0	102·8	105·9	88·0	479·0	19·8	65·5	84·7	114·5	46·0	46·0	48·0	20·8	10·7	20·0	30·6	4·1	8·9	39·8	39·8
119·0	139·0	107·0	104·8	119·7	91·0	489·0	16·8	65·6	92·7	117·0	51·1	49·2	48·6	22·6	9·6	20·3	32·8	4·4	11·9	39·5	39·1
124·0	117·0	118·0	107·0	105·8	97·4	493·0	16·2	65·0	84·5	114·5	47·9	48·1	47·3	22·0	10·2	17·5	29·3	4·2	8·8	39·7	37·8
126·0	112·0	118·5	108·5	101·4	100·4	493·0	23·2	73·4	87·4	114·0?	51·4	50·3	50·3	23·7	8·7	18·5	28·0	2·1	7·9	42·1	41·5
122·5	120·0	110·0	107·2	108·0	94·0	489·0	21·2	68·6	94·2	118·0	47·0	48·9	49·1	26·7	11·6	19·7	31·0	2·9	10·5	41·2	40·8
115·0	128·5	106·0	98·9	113·2	94·2	481·0	17·8	63·1	85·5	—	46·0	46·3	45·4	22·3	9·1	21·5	29·0	4·1	12·3	38·2	39·2
126·5	134·0	110·0	111·5	117·0	95·6	506·0	22·3	74·1	88·0	120·5	50·7	50·3	50·7	25·5	10·7	22·7	32·8	4·1	12·4	45·0	44·0
125·5	132·0	118·0	110·7	119·6	99·0	505·0	14·9	63·4	91·0	115·0	48·8	47·6	48·7	22·7	12·1	19·1	31·7	4·3	10·6	38·0	38·0
123·2	124·7	111·1	107·8	110·8	95·4	490·4	18·7	67·1	89·4	117·3	48·4	48·4	48·4	23·6	10·8	19·7	30·5	3·77	10·0	40·4	39·8
30	30	27	30	30	27	30	29	29	28	25	29	29	29	29	24	24	24	29	29	29	29

	Orbits							Palate and Profile					Foramen				
SC	O_1R	O_1L	$O_1'R$	$O_1'L$	O_2R	O_2L	*IOW*	G_1	G_1'	G_2	*EH*	*GL*	*fml*	*fmb*	100 *B*/*L*	100 *H'*/*L*	100 *B*/*H'*
4·1	41·7	42·2	37·9	37·8	31·2	31·3	97·3	—	—	41·2	—	—	—	—	76·7	—	—
8·1	42·8	42·8	41·8	42·0	33·0	32·6	99·5	53·0	49·3	37·9	14·8	97·1	37·7	30·6	74·4	74·4	100·0
0·3	45·8	46·4	43·4	42·6	35·6	35·3	105·4	56·8	52·3	42·8	16·8	101·4	36·8	28·5	73·9	74·7	98·9
9·5	41·2	42·8	39·0	40·0	32·2	32·0	95·6	52·2	47·1	42·2	14·0	100·8	35·1	28·7	75·5	71·3	106·0
9·3	37·8	40·2	36·8	38·6	32·0	34·1	94·5	58·3	52·6	40·0	13·1	99·0	34·5	31·6	81·4	80·8	100·7
1·4	43·2	41·0	41·8	39·0	33·3	34·0	91·2	51·0	46·0	39·0	13·0	93·3	36·6	27·9	76·7	71·7	107·1
—	45·0	—	—	—	33·4	31·8	104·0	—	—	—	—	89·0	37·0	32·2	74·9	72·8	102·9
—	41·0	—	40·0	—	30·0	30·9	94·6	56·8	49·6	36·2	10·0	99·0	—	28·0	76·0	79·2	96·1
0·3	46·0	44·7	—	—	34·0	34·9	101·2	52·9	46·8	—	—	97·0	36·6	29·4	74·5	80·4	92·6
0·8	42·0	40·6	39·0	39·0	34·0	33·2	95·7	51·2	45·8	34·8	10·2	94·4	34·5	28·2	76·6	76·6	100·0
9·1	40·2	39·8	38·8	38·7	33·7	34·5	90 6	50·1	44·2	39·8	9·2	90·4	37·5	31·7	79·2	79·7	99·3
7·0	42·0	40·2	38·7	37·9	33·8	34·2	94·5	51·0	46·8	38·8	12·8	93·3	33·6	30·5	78·7	77·0	102·2
1·0	40·2	39·6	36·2	36·3	34·5	35·5	87·0	48·8	44·2	34·5	11·0	95·0	33·0	29·7	76·9	76·6	100·4
7·0	40·3	40·4	38·2	38·9	34·4	34·8	95·4	57·0	52·8	41·0	19·2	98·6	32·9	30·0	78·0	81·1	96·2
7·4	43·0	40·2	41·6	37·8	35·0	35·6	93·4	48·2	43·6	40·6	11·8	91·2	36·2	29·8	80·9	78·3	103·3
0·2	43·2	43·2	40·6	40·0	33·2	34·0	100·6	55·8	51·0	40·8	14·2	102·7	38·0	31·7	73·4	73·4	100·0
2·7	41·8	43·0	40·2	39·4	32·4	31·7	99·4	60·0	53·0	45·6	15·2	104·8	36·0	34·0	69·8	73·0	95·7
9·4	47·8	47·2	46·5	45·4	33·8	33·6	102·1	51·8	47·0	39·3	15·6	100·0	37·7	32·4	71·5	75·0	95·3
0·6	42·8	42·8	39·6	39·2	36·1	36·1	96·2	49·0	43·8	—	—	91·6	35·2	31·0	78·9	76·7	102·9
7·4	44·7	45·8	42·3	43·2	36·8	36·8	100·8	51·2	46·3	41·0	16·3	100·8	41·0	32·5	74·2	78·1	95·0
1·4	—	38·8	—	—	30·2	30·8	90·2	—	—	—	—	—	—	—	74·7	—	—
9·5	43·8	42·2	41·2	40·8	31·5	32·7	95·3	53·6	49·1	43·0	10·8	101·2	32·0	29·0	76·8	73·5	104·6
3·0	46·2	46·0	43·2	42·2	37·4	37 0	108·2	52·0	47·2	40·0	11·2	108·0	37·5	30·5	73·8	74·1	99·7
6·7	38·6	38·4	35·2	35·2	30·8	31·0	85·0	46·0	42·0	39·7	14·2	85·7	36·2	28·0	80·4	78·4	102·6
9·1	41·6	41·6	38·8	39·0	35·2	33·2	97 0	53·5	47·6	43·0	13·0	100·8	33·0	27·3	78·1	71·7	108·9
7·9	40·8	40·0	38·8	38·8	36·5	36·4	93·0	47·0	43·5	38·8	13·2	91·4	36·1	30·1	75·4	74·9	100·7
8·7	39·9	39·8	38·1	38·2	29·2	29·3	93·0	50·2	44·1	40·4	9·7	95·8	33·8	29·7	74·7	74·2	100·7
8·5	43·9	43·8	41·0	40·4	33·0	34·2	96·0	50·8	47·6	—	—	96·3	38·1	31·0	71·2	72·6	98·1
9·9	43·0	44·0	38·6	40·0	34·0	33·2	95 0	44·6	41·8	38·6	14·8	94·4	36·2	31·7	73·8	72·5	101·9
9·9	40·0	41·0	36·4	37·4	31·2	32 0	90·4	48·8	43·8	37·3	10·3	90·7	36·4	30·6	75·6	76·8	98·5
1·0	43·2	42·0	40·6	39·8	34·8	35·2	98·6	55·0	47·4	39·0	13·5	98·5	37·2	28·3	77·7	76·9	101·1
8·2	40·0	40·0	36·0	35·6	30·4	32·2	92·7	47·6	43·8	43·0	15·4	90·0	41·5	32·1 ?	77·0	77·0	100·0
0·3	43·6	43·2	—	—	35·0	—	92·6	47·2	45·0	40·1	14·0	97·0	33·3	27·0	81·1	76·3	106·2
9·7	42·2	—	39·8	—	35·0	—	91·6	—	—	—	—	—	35·3	29·0 ?	76·3	74·6	102·3
9·6	39·8	39·4	37·6	36·8	31·8	31·8	90·0	51·2	44·8	40·2	12·7	92·8	33·5	29·4	74·3	73·1	101·6
9·8	42·6	41·2	41·0	40·0	33·0	33·5	100·4	49·9	47·8	—	—	91·5	36·5	30·5	71·6	68·8	104·1
7·5	45·2	43·0	42·1	40·0	35·4	35·7	99·0	49·0	45·5	42·0	12·2	97·0	40·2	30·8	79·2	80·9	97·9
8·2	—	39·3	—	37·1	33·4	33·3	88·0	45·3	42·6	—	—	89·0	33·2	30·1	76·2	75·7	100·7
—	41·3	—	40·0	—	31·2	—	91·4	53·7	47·5	—	—	94·6	34·0	29·5	78·0	73·7	105·8
8·0	41·1	41·6	39·6	39·8	32·0	33·0	95·0	53·1	48·8	41·0 ?	12·3 ?	96·6	34·2	29·1 ?	74·9	71·8	104·4
9·5	42·4	41·9	39·7	39·3	33·3	33·6	95·5	51·5	46·7	40·1	13·2	96·0	35·9	30·1	76·1	75·5	100·9
37	38	36	35	33	40	37	40	36	36	31	30	37	37	38	40	38	38
9·1	40·4	40·6	—	—	33·6	33·0	89·5	47·6	44·0	—	—	85·0	34·2	—	75·6	74·4	101·6
9·1	41·9	41·6	39·2	39·4	32·0	32·7	88·8	47·3	44·8	34·0	11·7	92·2	34·0	28·2	77·7	74·9	103·7
8·8	38·6	38·0	36·0	35·0	31·0	31·0	85·1	—	—	39·2	—	92·0	33·5	29·0	74·6	76·1	98·1
8·4	41·2	40·6	37·3	—	32·3	32·2	87·5	51·0	48·0	42·0	12·9	95·0	34·2	29·8	75·1	76·3	98·5
7·6	35·8	35·6	33·8	32·5	29·0	29·3	79·7	44·3	39·3	36·2	9·5	85·0	34·5	27·4	73·1	71·6	102·1
11·6	43·2	43·0	40·8	40·0	33·0	33·0	95·5	52·0	48·0	38·8	11·0	95·5	—	30·4	79·3	72·8	109·0
11·0	40·0	38·2	36·6	36·0	33·5	33·6	87·0	49·0	44·0	39·8	11·6	89·6	35·4	31·8	77·3	79·8	97·0
9·7	41·0	40·7	38·2	37·4	35·3	35·7	92·0	48·2	43·9	35·7	11·6	90·0	34·6	27·8	74·5	72·5	102·7
10·4	39·6	39·8	36·0	—	29·2	29·9	90·3	—	—	37·6	—	84·6	33·0	28·0	79·0	76·5	103·2
10·1	40·2	39·6	37·6	36·2	29·4	29·0	86·4	50·8	46·0	39·0	13·5	99·1	33·1	28·5	76·3	78·6	97·1
10·5	37·8	36·0	35·8	34·5	30·5	29·8	90·4	47·7	43·8	39·2	9·3	92·1	31·3	28·0	74·5	73·0	102·0
8·2	41·0	41·6	38·2	38·1	32·0	32·0	89·7	48·1	43·0	34·8	7·8	88·7	33·9	26·5	77·9	77·3	100·8
—	41·6	41·2	—	—	34·2	33·6	95·7	—	—	37·0	10·6	91·0 ?	—	26·8	78·6	70·5	111·5
8·0	39·4	39·0	38·2	37·7	35·0	33·0	89·5	45·2	41·7	35·3	13·0	87·2	32·8	25·5	81·2	78·9	103·0
12·5	41·3	41·0	37·9	37·8	35·3	35·4	88·7	46·2	40·2	36·8	14·3	81·6	35·3	28·9	83·7	76·7	109·1
8·4	38·2	37·8	36·0	35·1	33·2	33·8	84·5	45·2	41·8	36·8	9·8	87·3	31·9	30·0	80·9	78·4	103·1
9·7	41·7	43·0	39·4	40·0	33·4	33·8	94·1	41·2	38·2	35·9	9·0	81·7	32·8	24·7	80·6	75·7	106·5
11·3	39·0	38·0	36·0	35·4	28·8	30·0	86·4	—	—	38·6	—	97·8	33·4	29·5	74·3	76·0	97·7
9·6	41·8	—	37·2	—	33·0	—	94·4	—	44·8	40·3	13·4	88·0	33·8	26·2	75·9 ?	73·3 ?	103·5
12·8	37·0	37·7	35·2	35·2	31·9	32·7	91·9	51·7	45·8	—	—	97·4	29·6	26·8	75·1	79·2	94·9
10·2	41·2	40·0	37·2	36·2	33·0	34·0	93·0	44·0	42·8	38·8	14·8	82·6	—	28·7	77·0	80·9	95·1
9·1	43·0	42·0	—	—	34·8	34·9	96·9	43·8	40·2	39·2	13·6	88·3	35·4	34·8 ?	77·1	76·3	101·1
10·0	—	35·9	—	33·0	32·2	30·2	86·4	—	—	—	—	—	36·0	28·8	74·3	75·4	98·4
8·9	39·8	39·8	35·3	35·2	34·0	33·0	90·0	46·0	40·8	39·2	13·3	88·0	32·9	28·9	75·5	71·7	105·3
11·9	39·5	39·1	37·2	36·0	33·2	33·1	89·9	48·2	44·7	39·1	11·7	88·7	31·8	27·1	77·0	75·3	102·3
8·8	39·7	37·8	36·1	36·0	34·0	34·0	87·6	46·8	42·9	—	—	86·3	33·4	28·3	76·3	74·0	103·1
7·9	42·1	41·5	39·4	39·0	35·7	35·0	92·7	47·2	44·3	—	—	91·0	33·7	28·0	78·2	75·6	103·5
10·5	41·2	40·8	38·0	38·0	32·8	33·0	91·7	49·4	45·6	37·6	11·8	90·9	36·3	26·6	77·9	72·1	108·1
12·3	38·2	39·2	36·0	36·2	32·2	31·8	90·3	—	—	38·2	—	91·5	35·7	—	78·3	76·0	103·1
12·4	45·0	44·0	43·0	41·6	34·2	34·8	95·5	49·0	45·0	—	—	93·1	33·5	29·3	78·7	74·8	105·2
10·6	38·0	38·0	37·5	37·2	33·2	34·0	90·3	44·2	40·4	35·8	7·6	86·3	35·6	26·7	78·1	72·8	107·3
10·0	40·4	39·8	37·4	36·7	32·7	32·7	90·0	47·4	43·5	37·9	11·7	89·7	33·7	28·4	77·2	75·5	102·3
29	29	29	26	24	30	29	30	23	24	24	20	29	27	28	30	30	30

...ndividual and Mean Measurements of Ninth Dynasty Egyptian Skulls from Sedment.

Indices

	100 H'/L	100 B/H'	100 $\frac{B-H'}{L}$	100 H/L	100 B/H	*Oc. I.*	100 $\frac{G'H}{GB}$	100 $\frac{NB}{NH}$ $_R$	100 $\frac{NB}{NH}$ $_L$	100 $\frac{NB}{NH'}$	100 $\frac{DS}{DC}$	100 $\frac{SS}{SC}$	100 $\frac{O_2}{O_1}$ $_R$	100 $\frac{O_2}{O_1}$ $_L$	100 $\frac{O_2}{O_1'}$ $_R$	100 $\frac{O_2}{O_1'}$ $_L$	100 $\frac{G_2}{G_1}$	100 $\frac{G_2}{G_1'}$
	—	—	—	—	—	—	70·0	49·2	49·1	49·8	54·4	26·2	74·8	74·2	82·3	82·8	—	—
	74·4	100·0	0·0	74·7	99·6	64·48	80·0	45·9	45·8	44·1	57·4	43·2	77·1	76·2	78·9	77·6	71·5	76·9
	74·7	98·9	−0·8	74·7	98·9	63·87	74·8	45·0	45·1	45·5	51·7	49·5	77·7	76·1	82·0	82·9	75·4	81·8
	71·3	106·0	4·3	72·6	104·0	56·83	75·5	46·4	45·6	45·9	51·3	26·3	78·2	74·8	82·6	80·0	80·8	89·6
	80·8	100·7	0·6	81·4	100·0	62·65+	81·5	40·7	41·0	39·1	56·3	36·6	84·7	84·8	87·0	88·3	68·6	76·0
	71·7	107·1	5·1	72·5	106·1	68·18	81·9	41·7	41·4	41·6	69·1	46·5	77·1	82·9	79·7	87·2	76·5	84·8
	72·8	102·9	2·1	73·4	102·2	65·96	—	—	—	—	—	—	74·2	—	—	—	—	—
	79·2	96·1	−3·1	80·0	95·1	—	77·5	47·9	48·0	51·1	—	—	73·2	—	75·0	—	63·7	73·0
	80·4	92·6	−5·9	80·9	92·0	61·77	76·5	49·4	47·7	49·3	—	42·7	73·9	78·1	—	—	—	—
	76·6	100·0	0·0	76·6	100·0	61·58	72·7	43·5	43·6	43·7	51·8	33·3	81·0	81·8	87·1	85·1	68·0	76·0
	79·7	99·3	−0·6	79·7	99·3	73·42	80·3	47·1	47·0	46·6	46·9	25·3	83·8	86·7	86·9	89·1	79·4	90·0
	77·0	102·2	1·7	77·5	101·4	62·15	78·9	48·0	48·4	49·1	56·9	24·3	80·5	85·1	87·3	90·2	76·1	82·9
	76·6	100·4	0·3	76·3	100·7	66·71	85·6	37·9	37·5	39·9	51·9	33·6	85·8	89·6	95·3	97·8	70·7	78·1
	81·1	96·2	−3·1	81·4	95·8	65·26	77·9	43·8	43·9	43·6	65·8	28·6	85·4	86·1	90·1	89·5	71·9	77·7
	78·3	103·3	2·6	78·8	102·6	60·46	75·9	46·5	46·2	48·5	59·9	56·8	81·4	88·6	84·1	94·2	84·2	93·1
	73·4	100·0	0·0	73·4	100·0	60·38	77·7	49·8	49·9	51·5	61·0	45·1	76·9	78·7	81·8	85·0	73·1	80·0
	73·0	95·7	−3·2	73·0	95·7	62·25−	71·9	46·3	46·5	49·2	58·6	41·7	77·5	73·7	80·6	80·5	76·0	86·0
	75·0	95·3	−3·5	75·8	94·3	59·36	72·0	49·6	49·9	51·9	53·5	61·7	70·7	71·2	72·7	74·0	75·9	83·6
	76·7	102·9	2·2	77·5	101·8	61·31	74·9	46·8	46·8	47·9	47·7	39·6	84·3	84·3	91·2	92·1	—	—
	78·1	95·0	−3·9	78·7	94·4	70·87	75·2	44·5	45·0	43·6	63·3	55·4	82·3	80·4	87·0	85·2	79·9	88·6
	—	—	—	—	—	—	—	50·7	50·5	49·8	—	39·5	—	79·4	—	—	—	—
	73·5	104·6	3·4	73·5	104·6	57·34	79·0	53·5	52·1	56·9	59·4	36·8	71·9	77·5	76·5	80·1	80·2	87·6
	74·1	99·7	−0·3	74·4	99·3	58·52	79·4	49·0	48·9	49·6	55·8	43·1	81·0	80·4	86·6	87·7	76·9	84·7
	78·4	102·6	2·0	78·4	102·6	68·01	71·7	49·5	49·2	50·4	52·6	25·4	79·8	80·7	87·5	88·1	86·3	94·5
	71·7	108·9	6·4	72·2	108·1	63·30	76·6	49·3	48·3	49·1	55·8	48·4	84·6	79·8	90·7	85·1	80·4	90·3
	74·9	100·7	0·5	75·4	100·0	62·65	80·3	44·5	45·9	44·1	58·5	59·5	89·5	91·0	94·1	93·8	82·6	89·2
	74·2	100·7	0·6	74·7	100·0	65·13	74·2	56·2	55·4	58·8	42·7	33·3	73·2	73·6	76·6	76·7	80·5	91·6
	72·6	98·1	−1·4	73·9	96·3	62·15−	77·0	47·4	47·4	48·2	71·4	56·5	75·2	78·1	80·5	84·7	—	—
	72·5	101·9	1·4	72·5	101·9	61·22	84·7	46·1	45·6	45·5	54·4	31·3	79·1	75·5	88·1	83·0	86·5	92·3
	76·8	98·5	−1·1	77·1	93·1	81·80	70·1	53·2	52·8	53·8	66·1	46·5	78·0	78·0	85·7	85·6	76·4	85·2
	76·9	101·1	0·8	77·2	100·7	58·83	73·5	47·3	47·3	48·0	52·4	48·2	80·6	83·8	85·7	88·4	70·9	82·3
	77·0	100·0	0·0	77·5	99·3	74·74	74·2	46·4	46·5	46·4	63·8	48·8	76·0	80·5	84·4	90·4	90·3	98·2
	76·3	106·2	4·7	76·6	105·8	60·63	72·8	45·2	45·0	44·4	—	41·7	80·3	—	—	—	85·0	89·1
	74·6	102·3	1·7	74·9	101·9	60·55+	—	—	—	—	—	24·7	82·9	—	87·9	—	—	—
	73·1	101·6	1·1	74·3	100·0	65·91	77·7	48·3	48·4	47·4	47·9	36·5	79·9	80·7	84·6	86·4	78·5	89·7
	68·8	104·1	2·8	69·6	103·0	65·13	71·1	52·8	51·8	53·0	67·3	45·9	77·5	81·3	80·5	83·8	—	—
	80·9	97·9	−1·7	82·0	96·6	65·26	72·9	43·2	43·5	43·2	46·6?	46·7	78·3	83·0	84·1	89·3	85·7	92·3
	75·7	100·7	0·5	75·4	101·1	65·82	—	46·0	47·1	45·4	—	56·1	—	84·7	—	89·8	—	—
	73·7	105·8	4·2	74·8	104·3	65·26	77·6	51·1?	50·7?	50·9?	—	—	75·5	—	78·0	—	—	—
	71·8	104·4	3·2	72·3	103·6	59·49	74·5	47·0	47·1	47·3	48·0	33·8	77·9	79·3	80·8	82·9	77·2?	84·0?
	75·5	100·9	+0·6	75·9	100·3	64·03	76·3	47·3	47·2	47·7	56·3	41·1	79·0	80·6	84·1	86·0	77·6	85·6
	38	38	38	38	38	37	36	38	38	38	32	37	38	35	35	33	30	30
	74·4	101·6	1·2	75·0	100·8	61·67	79·5	51·2	51·2	51·8	—	37·4	83·2	81·3	—	—	—	—
	74·9	103·7	2·8	76·0	102·2	62·55+	76·3	46·8	46·8	46·7	63·7	33·0	76·4	78·6	81·6	83·0	71·9	75·9
	76·1	98·1	−1·4	77·2	96·6	63·87	75·1	45·5	45·5	45·4	56·9	33·0	80·3	81·6	86·1	88·6	—	—
	76·3	98·5	−1·2	77·5	97·0	64·61	75·8	43·3	43·3	41·3	—	50·0	78·4	79·3	86·6	—	82·4	87·5
	71·6	102·1	1·5	71·9	101·7	63·76	67·5	55·1	55·2	55·2	45·1	26·3	81·0	82·3	85·8	90·2	81·7	92·1
	72·8	109·0	6·5	73·6	107·7	—	77·2	42·7	42·4	42·9	54·9	38·8	76·4	76·7	80·9	82·5	74·6	80·8
	79·8	97·0	−2·4	80·4	96·2	64·87	76·0	50·4	49·8	51·1	68·8	46·4	83·8	88·0	91·5	93·3	81·2	90·5
	72·5	102·7	2·0	73·1	101·9	59·92	76·0	42·7	42·9	42·2	62·1	49·5	86·1	87·7	92·4	95·5	74·1	81·3
	76·5	103·2	2·4	78·0	101·2	66·40	69·2	57·8	57·2	60·8	—	44·2	73·7	75·1	81·1	—	—	—
	78·6	97·1	−2·3	79·8	95·7	64·87	69·6	54·9	54·4	55·5	50·0	33·7	73·1	73·2	78·2	80·1	77·2	84·8
	73·0	102·0	1·5	73·3	101·6	62·05+	72·3	50·2	49·3	49·7	52·3	47·6	80·7	82·8	85·2	86·4	82·2	89·5
	77·3	100·8	0·6	77·9	100·0	62·55+	71·1	49·5	49·4	49·6	57·4	24·4	78·0	76·9	83·8	84·0	72·3	80·9
	70·5	111·5	8·1	70·7	111·1	—	74·4?	50·9	50·1	51·9	—	—	82·2	81·6	—	—	—	—
	78·9	103·0	2·3	79·5	102·2	61·77	77·0	45·7	45·8	44·9	64·6	38·8	88·8	84·6	91·6	87·5	78·1	84·7
	76·7	109·1	7·0	76·8	108·7	60·07	74·5	47·3	47·5	47·2	51·2	34·4	85·5	86·3	93·1	93·7	79·7	91·5
	78·4	103·1	2·5	78·7	102·7	65·13	74·4	44·7	45·3	44·9	67·7	53·6	86·9	89·4	92·2	96·3	81·4	88·0
	75·7	106·5	4·9	75·7	106·5	67·18	77·6	45·9	46·7	45·9	61·0	28·9	80·1	78·6	84·8	84·5	87·1	94·0
	76·0	97·7	−1·7	77·1	96·3	63·76	73·5	53·3	54·5	53·2	50·0	36·3	73·8	78·9	80·0	84·7	—	—
	73·3?	103·5	2·6?	73·9?	102·7	59·92	—	46·5	46·0	46·5	—	39·6	78·9	—	88·7	—	—	90·0
	79·2	94·9	−4·0	79·2	94·9	66·86	67·6	55·8	56·5	55·4	47·6	32·8	86·2	86·7	90·6	92·9	—	—
	80·9	95·1	−3·9	81·5	94·5	—	79·4	51·4	51·8	50·7	55·0	42·2	80·1	85·9	88·7	93·9	88·2	90·7
	76·3	101·1	0·8	76·8	100·4	67·84	73·4	46·5	47·7	49·0	—	35·2	80·9	83·1	—	—	89·5	97·5
	75·4	98·4	−1·2	75·7	98·1	63·99	—	—	—	—	58·5	41·0	—	84·1	—	91·5	—	—
	71·7	105·3	3·8	72·6	104·1	62·65+	77·3	45·2	43·3	45·2	53·5	46·1	85·4	82·9	96·3	93·8	85·2	96·1
	75·3	102·3	1·7	76·4	100·8	62·05+	70·8	45·9	46·5	44·2	47·3	37·0	84·1	84·7	89·2	92·0	81·1	87·5
	74·0	103·1	2·3	74·9	101·9	59·22	76·9	45·7	46·5	45·9	58·3	47·7	85·6	80·9	94·2	94·4	—	—
	75·6	103·5	2·6	75·9	103·1	61·67	84·0	47·1	47·1	46·1	47·0	26·6	84·8	84·3	90·6	89·7	—	—
	72·1	108·1	5·8	72·7	107·2	62·65+	72·8	54·6	54·4	56·8	58·9	27·6	79·6	80·9	86·3	86·9	76·1	82·5
	76·0	103·1	2·4	77·7	100·8	68·89	73·8	48·2	49·1	48·5	42·3	33·3	84·3	81·1	89·4	87·8	—	—
	74·8	105·2	3·9	75·4	104·5	64·87	84·2	50·7	50·3	50·3	47·1	33·1	76·0	79·1	79·5	83·7	—	—
	72·8	107·3	5·3	73·6	106·0	60·63	69·7	47·7	46·6	46·5	63·4	40·6	87·4	89·5	88·5	91·4	81·0	88·6
	75·5	102·3	+1·7	76·2	101·4	63·54	74·9	48·8	48·8	48·9	55·1	37·9	81·2	82·2	87·3	89·0	80·2	87·7
	30	30	30	30	30	27	28	29	29	29	24	29	29	29	26	24	18	19

				Angles in degrees						Capacity	
$100\frac{G_2}{G_1}$	$100\frac{G_2}{G_1'}$	$100\frac{EH}{EB}$	$100\frac{fmb}{fml}$	$N\angle$	$A\angle$	$B\angle$	θ_1	θ_2	$P\angle$	C	
—	—	—	—	—	—	—	—	—	—	—	Base and left temporal bone l
71·5	76·9	39·1	81·2	63·2	74·6	42·2	32·1	10·1	84·7	1496·9	Ageing, coronal and sagittal s
75·4	81·8	39·3	77·4	63·0	74·6	42·4	31·5	10·9	85·5	1478·0	Aged, all sutures nearly oblite
80·8	89·6	33·2	81·8	68·2	68·9	42·9	27·6	15·3	84·2	1416·2	Base partly defective; R rami
68·6	76·0	32·8	91·6	68·9	67·8	43·3	35·1	16·8	76·0	1341·3	Coronal suture beginning to cl and lower L 1st molar; tyn
76·5	84·8	33·3	76·2	64·0	72·0	44·0	27·2	8·2	88·8	—	Parts of zygomatic arches and worn; tym perforation
—	—	—	87·0	—	—	—	—	—	—	1519·1	Zygomatic arches and palate p
63·7	73·0	27·6	—	68·0	69·8	42·2	29·8	12·4	82·2	—	Base, temporal and L maxilla
—	—	—	80·3	61·4	76·7	41·9	35·1	6·8	83·5	1575·0	Left mandible missing; coron and premolars lost from u
68·0	76·0	29·3	81·7	61·0	80·0	39·0	30·8	8·2	88·2	1410·3	Coronal suture closed, sagittal
79·4	90·0	23·1	84·5	62·6	70·9	46·5	34·6	11·9	82·8	1368·6	Ageing, coronal and sagittal s
76·1	82·9	33·0	90·8	67·0	67·6	45·4	33·9	11·5	79·1	1365·4	No mandible; coronal suture f supra-occipital; JL.
70·7	78·1	31·9	90·0	61·8	74·1	44·1	33·2	10·9	85·0	1462·1	R ramus of mandible missing;
71·9	77·7	46·8	91·2	63·1	71·2	45·7	33·9	11·8	83·0	1439·4	Coronal suture beginning to c
84·2	93·1	29·1	82·3	61·3	77·2	41·5	31·7	9·8	87·0	1307·4	Base partly missing; L coronal from upper jaw and 3rd M
73·1	80·0	34·8	83·4	67·1	69·9	43·0	28·1	14·9	84·8	1500·8	Ageing, coronal suture beginn
76·0	86·0	33·3	94·4	72·1	67·6	40·3	27·1	13·2	80·8	1433·8	Complex sutures; single worm
75·9	83·6	39·7	85·9	72·1	67·7	40·2	22·5	17·7	85·4	1440·9	Ageing, coronal and sagittal s
—	—	—	88·1	60·0	78·6	41·4	35·6	5·8	84·4	1265·8	All sutures beginning to close
79·9	88·6	39·8	79·3	65·1	71·4	43·5	32·9	10·6	82·0	1447·4	Aged, all sutures almost oblite
—	—	—	—	—	—	—	—	—	90·1	—	Base and right side of face an
80·2	87·6	25·1	90·6	67·5	69·1	43·4	31·5	11·9	81·0	1218·9	No mandible; teeth complete
76·9	84·7	28·0	81·3	67·2	69·0	43·8	25·8	18·0	87·0	1487·8	Aged, all sutures partly oblite
86·3	94·5	35·8	77·3	61·8	78·6	39·6	28·4	11·2	89·8	1415·5	Ageing, coronal suture closed,
80·4	90·3	30·2	82·7	70·3	66·0	43·7	26·8	16·9	82·9	1452·0	Coronal and sagittal sutures c
82·6	89·2	34·0	83·4	62·7	70·1	47·2	33·3	13·9	84·0	1498·9	No teeth lost from either jaw
80·5	91·6	24·0	87·9	65·8	75·5	38·7	28·0	10·7	86·2	1371·3	Zygomatic arches partly miss
—	—	—	81·4	62·8	74·3	42·9	31·9	11·0	85·3	—	Base and L parietal bone par
86·5	92·3	38·3	87·6	61·9	73·3	44·8	34·1	10·8	84·0	1397·9	No mandible; aged, all suture
76·4	85·2	27·7	84·1	67·9	73·1	39·0	27·0	12·0	85·1	1312·6	R ramus of mandible missing
70·9	82·3	34·6	76·1	70·1	68·9	41·0	33·3	7·7	76·6	1484·5	Wormians in lambdoid suture
90·3	98·2	35·8	77·3?	63·0	74·7	42·3	29·1	13·2	87·9	1467·6	Complex sutures; wormians in
85·0	89·1	34·9	81·1	66·6	73·5	39·9	30·3	9·6	83·1	1375·8?	L face partly defective; ageir
—	—	—	82·2?	—	—	—	—	—	—	1325·7	Greater part of face missing; c tions R and L.
78·5	89·7	31·6	87·8	65·4	72·5	42·1	26·8	15·3	87·8	1173·3	Young adult, basal suture ope
—	—	—	83·6	63·4	72·0	44·6	29·5	15·1	87·1	1504·1	Occipital bone partly defective suture; teeth of both jaws incisure L.
85·7	92·3	29·0	76·6	63·7	72·5	43·8	36·1	7·7	80·2	1489·1	No mandible; aged, all suture
—	—	—	90·7	62·8	76·2	41·0	30·4	10·6	86·8	1537·0	R side facial skeleton partly d before death but none fro
—	—	—	86·8	64·7	73·3	42·0	30·7	11·3	84·6	1550·3	L side face defective; R side r complete; JL.
77·2?	84·0?	30·0?	85·1?	66·4	72·1	41·5	27·4	14·1	86·2	1609·6	Aged, all sutures almost oblite
77·6	85·6	32·8	84·1	65°·1	72°·4	42°·5	30°·6	11°·9	84°·4	1426·9[m]	
30	30	30	37	36	36	36	36	36	37	35	
—	—	—	—	60·8	73·9	45·3	36·2	9·1	83·0	1138·2	Aged, coronal and sagittal sut
71·9	75·9	34·4	82·9	65·9	72·1	42·0	27·8	14·2	86·3	1372·6	R ramus of mandible missing R and L; J —.
—	—	—	86·6	64·8	75·1	40·1	33·2	6·9	82·0	—	No mandible; L side brain bo
82·4	87·5	30·7	87·1	66·3	71·8	41·9	30·3	11·6	83·4	1246·2	L ramus of mandible missing
81·7	92·1	26·2	79·4	66·3	76·9	36·8	25·7	11·1	88·0	1101·0?	No mandible; ossicle of sagitt
74·6	80·8	28·4	—	62·3	75·0	42·7	27·9	14·8	89·8	—	Occiput and base largely defe
81·2	90·5	29·1	89·8	65·2	72·0	42·8	33·2	9·6	81·6	1205·9	No teeth lost before death, no
74·1	81·3	32·5	80·3	62·6	72·4	45·0	32·1	12·9	85·3	1309·4	Young adult, basal suture not
—	—	—	84·8	65·2	73·4	41·4	34·0	7·4	80·8	1155·1	Young adult, wormians in lar
77·2	84·8	34·6	86·1	72·4	69·2	38·4	30·9	7·5	76·7	—	Base partly defective; wormi
82·2	89·5	23·7	89·5	67·2	73·2	39·6	27·8	11·8	85·0	1147·9	Wormians in lambdoid suture R and L.
72·3	80·9	22·4	78·2	65·8	72·4	41·8	29·2	12·6	85·0	1263·8	L ramus of mandible missing small exostosis on supra-oc
—	—	28·6	—	68·4?	68·2?	43·4?	23·8?	10·6?	87·8	—	Base partly defective; aged, a
78·1	84·7	36·8	77·7	61·6	72·0	46·4	29·5	16·9	88·9	1296·4	Mandible partly broken; all s
79·7	91·5	38·9	81·9	63·2	69·9	46·9	33·6	13·3	83·2	1474·1	Coronal suture closing and ot
81·4	88·0	26·6	94·0	61·2	77·4	41·4	35·2	6·2	83·6	1117·3	Young adult; small ossicles in border; epipteric bones R.
87·1	94·0	25·1	75·3	61·2	73·3	45·5	31·4	14·1	87·4	1289·9	Aged, coronal and sagittal su
—	—	—	88·3	69·1	72·3	38·6	28·1	10·3	82·8	1174·0	L coronal suture beginning to
—	90·0	33·3	77·5	63·6	69·8	46·6	30·0	16·6	86·4	—	L facial skeleton partly defect death (?), teeth extremely
—	—	—	90·5	68·4	72·1	39·5	29·4	10·1	82·2	1160·3	No mandible; coronal sutures
88·2	90·7	38·1	—	57·4	76·7	45·9	—	—	—	1426·6	Base partly defective; young
89·5	97·5	34·7	98·3?	60·1	76·7	43·2	34·5	8·7	85·4	1453·3?	Ageing, coronal suture closed,
—	—	—	80·0	—	—	—	—	—	—	1250·1	Facial bones partly missing;
85·2	96·1	33·9	87·8	65·0	72·7	42·3	29·4	12·9	85·6	1183·1	Single wormians in L coronal
81·1	87·5	29·9	85·2	63·2	75·5	41·3	31·7	9·6	85·1	1295·1	No mandible; ossicles of lamb
—	—	—	84·7	64·5	72·7	42·8	33·4	9·4	82·1	1265·8	L ramus of mandible incompl
—	—	—	83·1	61·2	73·8	45·0	35·6	9·4	83·2	1221·5	Lower jaw and R side face in
76·1	82·5	31·4	73·3	67·0	69·0	44·0	28·4	15·6	84·6	1187·6	Ageing, coronal and sagittal s
—	—	—	—	66·2	74·6	39·2	30·8?	8·4?	83·0?	1289·9	Brain box partly defective; o
—	—	—	87·5	63·6	70·9	45·5	30·2	15·3	86·2	1286·6	No mandible; coronal suture
81·0	88·6	21·2	75·0	64·2	74·2	41·6	29·9	11·7	85·9	1423·3	Juvenile; single wormian in L
80·2	87·7	31·0	84·4	64°·5	72°·9	42°·6	30°·8	11°·6	84°·4	1252·5[co]	
18	19	20	25	29	29	29	28	28	28	25	

Capacity C	Remarks
—	Base and left temporal bone largely defective; aged, coronal and sagittal sutures nearly obliterated, lambdoid suture closed;
1496·9	Ageing, coronal and sagittal sutures closed and partly obliterated, lambdoid suture beginning to close; teeth complete but w
1478·0	Aged, all sutures nearly obliterated; complete dentition and moderately worn; small precondyle R; JR; JL divided; whole
1416·2	Base partly defective; R ramus of mandible missing; ageing, coronal and sagittal sutures closed and nearly obliterated, lam
1341·3	Coronal suture beginning to close, sagittal suture obliterated at obelion, lambdoid suture beginning to close; ossicle at R aste and lower L 1st molar; tympanic perforations R and L; JR and JL divided.
●	Parts of zygomatic arches and R condyle of mandible missing; aged, coronal and sagittal sutures almost obliterated, lambdo worn; tympanic perforation R; JR; excrescence behind left condyle.
1519·1	Zygomatic arches and palate partly missing; aged, all sutures nearly obliterated; all teeth probably lost from upper jaw before
—	Base, temporal and L maxilla partly defective; L ramus of mandible missing; aged, all sutures nearly obliterated; ossicle at R
1575·0	Left mandible missing; coronal and sagittal sutures beginning to close, lambdoid suture clearly open; numerous wormians in and premolars lost from upper jaw, no teeth lost from R side of mandible, teeth considerably worn; epipteric bone R; JR
1410·3	Coronal suture closed, sagittal and lambdoid sutures beginning to close; ossicle of lambda, large ossicles in lambdoid suture
1368·6	Ageing, coronal and sagittal sutures closed, lambdoid suture beginning to close; teeth of both jaws complete and moderately
1365·4	No mandible; coronal suture beginning to close, other sutures open; ossicles in lambdoid suture R and L; teeth complete in supra-occipital; JL.
1462·4	R ramus of mandible missing; ossicle of lambdoid suture L; metopic suture RF + LP 5·3 mm.; all teeth in situ and slightly w
1439·6	Coronal suture beginning to close, sagittal suture closed, lambdoid suture open; perfect dentition and slightly worn; JL.
1307·4	Base partly missing; L coronal suture obliterated, R coronal and sagittal sutures beginning to close, lambdoid suture open; o from upper jaw and 3rd M's unworn, no 3rd M's in lower jaw, teeth moderately worn; epipteric bone L; asymmetrica
1500·8	Ageing, coronal suture beginning to close, sagittal suture nearly obliterated, lambdoid suture closed; no teeth lost before d
1433·8	Complex sutures; single wormian in lambdoid suture R and L; teeth complete and slightly worn; epipteric bones R and L;
1440·9	Ageing, coronal and sagittal sutures closed and partly obliterated, lambdoid suture beginning to close; a few molars and pre
1265·8	All sutures beginning to close; about 6 molar and premolars lost from upper jaw before death, no teeth lost from lower ja
1447·4	Aged, all sutures almost obliterated; no teeth lost from either jaw, teeth very worn; JL.
—	Base and right side of face and mandible partly defective; all sutures beginning to close; complex sutures; wormians in la
1218·9	No mandible; teeth complete and very worn; epipteric bones R and L; JR.
1487·8	Aged, all sutures partly obliterated; no 3rd M's in either jaw, upper L 1st molar lost before death and socket diseased, R 1
1415·5	Ageing, coronal suture closed, sagittal suture nearly obliterated, lambdoid suture beginning to close; no teeth lost from eit
1452·0	Coronal and sagittal sutures complex and nearly closed; perfect dentition, teeth slightly worn; epipteric bone R; J =.
1498·9	No teeth lost from either jaw before death, teeth moderately worn but 3rd M's less worn than other molars; precondyles
1371·3	Zygomatic arches partly missing; aged, all sutures closed and nearly obliterated; several teeth lost from both jaws, teeth
—	Base and L parietal bone partly defective; aged, all sutures almost obliterated; some teeth lost before death; epipteric b
1397·9	No mandible; aged, all sutures beginning to close; ossicles of lambdoid suture R and L; metopic suture RF + LP 2·2 mm
1312·6	R ramus of mandible missing; ageing, all sutures closed and partly obliterated; depression at obelion; I_2L and CL lost fr
1484·5	Wormians in lambdoid suture R and L; teeth of both jaws complete and slightly worn, one carious; JR.
1467·6	Complex sutures; wormians in lambdoid suture R and L; no teeth lost before death, only one 3rd molar (lower R), teeth m
1375·8?	L face partly defective; ageing, coronal and sagittal sutures nearly obliterated, lambdoid suture open; small wormians in
1325·7	Greater part of face missing; coronal suture beginning to close, other sutures open; a large ossicle in lower corner of L pariet tions R and L.
1173·3	Young adult, basal suture open, complete dentition; all 3rd molars erupted but unworn; wormian in lambdoid suture L; JI
1504·1	Occipital bone partly defective; aged, coronal suture closed, sagittal suture completely obliterated, lambdoid suture closed suture; teeth of both jaws defective, about 5 or 6 teeth lost from upper jaw and 4 or 5 teeth lost from lower jaw; JR; se incisure L.
1489·1	No mandible; aged, all sutures closed and nearly obliterated; no teeth lost before death, teeth considerably worn; JR.
1537·9	R side facial skeleton partly defective; coronal suture obliterated near pterion R and L but open above, sagittal and lambdo before death but none from lower, teeth markedly worn; JR; small blunt spine projecting posterior to basion.
1550·3	L side face defective; R side mandible partly missing; ageing, coronal and sagittal sutures closing, lambdoid suture open; complete; JL.
1609·6	Aged, all sutures almost obliterated, thinning in middle part of parietal bones due to age; possibly 3 molars lost from both j
1426·9[cc] 35	
1138·2	Aged, coronal and sagittal sutures almost obliterated, lambdoid suture closed; teeth of upper jaw L and lower jaw R most
1372·6	R ramus of mandible missing; ageing, coronal suture closed, sagittal suture nearly closed, lambdoid suture beginning R and L; J =.
—	No mandible; L side brain box and palate partly defective; wormians in lambdoid suture R and L, small ossicle above lam
1246·2	L ramus of mandible missing; wormians in lambdoid suture R and L, single wormian in sagittal suture above lambda; n
1101·0?	No mandible; ossicle of sagittal suture above lambda; wormians in lambdoid suture R and L; no teeth lost before death; tyn
—	Occiput and base largely defective; large epipteric bones R and L; teeth complete in upper jaw but very worn, L 3rd M ar
1205·9	No teeth lost before death, no 3rd M's in either jaw, teeth moderately worn; epipteric bones R and L; J =.
1309·4	Young adult, basal suture not completely closed; no teeth lost from either jaw; JR.
1155·1	Young adult, wormians in lambdoid sutures R and L; ossicle of lambda; teeth complete, 3rd M's not fully erupted; large
—	Base partly defective; wormians in lambdoid suture R and L; ossicle of L coronal suture above pterion; teeth complete a
1147·9	Wormians in lambdoid suture R and L; no 3rd M's in either jaw, teeth worn, upper P_3R lost before death; epipteric bones R and L.
1263·8	L ramus of mandible missing; L coronal suture closing but R open; wormians in lambdoid suture R and L; 3 or 4 teeth small exostosis on supra-occipital.
—	Base partly defective; aged, all sutures closed and nearly obliterated; no teeth of either jaw lost, teeth exceedingly worn.
1296·4	Mandible partly broken; all sutures closing; teeth complete but moderately worn; epipteric bones R and L; tympanic perfo
1474·1	Coronal suture closing and other sutures open; no teeth lost before death, no 3rd M's in either jaw, teeth moderately worn
1117·3	Young adult; small ossicles in middle parts of coronal suture R and L; small wormians in lambdoid suture R and L; basal border; epipteric bones R and L; JR.
1289·9	Aged, coronal and sagittal sutures nearly obliterated, lambdoid suture closing; several teeth lost from upper jaw, teeth of
1174·0	L coronal suture beginning to close and others open; perfect dentition but considerably worn; epipteric bone L; tympanic
—	L facial skeleton partly defective; mandible broken into two pieces; coronal suture closed and mostly obliterated, sagittal death (?), teeth extremely worn; epipteric bone L; JR; trace of suture between ex- and supra-occipital bone R.
1160·3	No mandible; coronal sutures closing and others open; wormians in lambdoid suture R and L; no teeth lost before death
1426·6	Base partly defective; young adult, basal suture probably open; teeth complete and slightly worn, 3rd M's not fully eru
1453·3?	Ageing, coronal suture closed, sagittal suture nearly closed, lambdoid suture open; wormians in lambdoid suture R and L;
1250·1	Facial bones partly missing; young adult, basal suture not completely closed; ossicle at R asterion; probably no teeth lost
1183·1	Single wormians in L coronal and R lambdoid sutures; teeth of both jaws complete and hardly worn; epipteric bone R; J
1295·1	No mandible; ossicles of lambda and in lambdoid suture L; traces of interparietal sutures R and L; teeth complete and ve
1265·8	L ramus of mandible incomplete; 3 wormians in R lambdoid suture at asterion; no teeth lost before death, no lower 3rd M
1221·5	Lower jaw and R side face incomplete; aged, all sutures closed, sagittal and coronal sutures partly obliterated; wormians i
1187·6	Ageing, coronal and sagittal sutures nearly obliterated, lambdoid suture open; no teeth lost before death, no 3rd M's in b
1289·9	Brain box partly defective; ossicle of lambda; metopic suture LF + RP 7·0 mm.; teeth of both jaws complete (?); JR; pr
1286·6	No mandible; coronal suture closed and almost obliterated, other sutures open; several teeth lost before death; tympanic
1423·3	Juvenile; single wormian in L lambdoid suture; basal suture wide open; 3rd M's not erupted in either jaw, teeth unworn;
1252·5[cc] 25	

	Remarks
	doid suture closed; no teeth lost from either jaw before death, teeth considerably worn.
	eeth complete but worn; epipteric bone R; JR.
	JL divided; whole skull unusually strong and massive.
	dy obliterated, lambdoid suture beginning to close; ossicles at asteria R and L; teeth complete but considerably worn; JR.
	se; ossicle at R asterion; perfect dentition but greatly worn; bone eroded by disease (?) at roots of both upper 1st molars
	obliterated, lambdoid suture closed; 2 1st molars lost from upper jaw, 2 incisors lost from lower jaw, teeth considerably
	n upper jaw before death, alveoli absorbed, teeth complete in lower jaw but exceedingly worn; large epipteric bone R; JR.
	erated; ossicle at R asterion; upper 3rd M's lost before death, teeth exceedingly worn; 2 small precondyles; JR.
	nerous wormians in lambdoid suture R and L; ossicle of lambda and a large ossicle in sagittal suture above it; 8 molars
	ipteric bone R; JR; massive inionic protuberance.
	n lambdoid suture R and L; teeth very worn, 2 1st molars lost from upper jaw, R 1st molar lost from lower jaw; J =.
	te and moderately worn; JR; JL divided.
	; teeth complete in upper jaw and slightly worn; tympanic perforation R; faint trace of metopic suture; protruding
	situ and slightly worn; JR.
	ightly worn; JL.
	doid suture open; ossicle in sagittal suture immediately above lambda; single wormian in L lambdoid suture; no teeth lost
	ne L; asymmetrical.
	teeth lost before death, teeth very worn; precondyle L; JR.
	ic bones R and L; JR; negroid?
	few molars and premolars probably lost from both jaws, teeth greatly worn; single precondyle; JR.
	lost from lower jaw, teeth considerably worn; JL.
	es; wormians in lambdoid suture R and L; teeth complete (?) and very worn.
	ocket diseased, R 1st M lost from lower jaw; 2 small precondyles; JR; large and strong skull.
	teeth lost from either jaw but all considerably worn; tympanic perforations R and L; J =.
	bone R; J =.
	olars; precondyles large L and small R; JR.
	n both jaws, teeth greatly worn; JR.
	death; epipteric bones R and L.
	e RF+LP 2·2 mm.; no teeth lost from upper jaw but teeth considerably worn; JR.
	I_2L and CL lost from upper jaw, 3rd ML lost from lower jaw, teeth greatly worn; JR; JL divided.
	(lower R), teeth moderately worn; JL.
	small wormians in lambdoid suture; teeth complete and considerably worn; JR.
	r corner of L parietal bone above asterion and mastoid; teeth complete in lower jaw but very worn; tympanic perfora-
	bdoid suture L; JL.
	doid suture closed and nearly obliterated; small wormians in coronal and lambdoid sutures; trace of transverse occipital
	lower jaw; JR; several palate bridges; markedly retreating frontal bone and prominent superciliary ridge; basi-occipital
	bly worn; JR.
	gittal and lambdoid sutures open; very small wormians in lambdoid suture R and L; 4 or 5 teeth lost from upper jaw
	to basion.
	doid suture open; ossicle of lambda, small ossicles in lambdoid suture; several teeth lost from upper jaw, lower teeth
	rs lost from both jaws and one molar carious, teeth very worn; large epipteric bone L; JL.
	lower jaw R mostly lost before death; tympanic perforation L; JL.
	suture beginning to close; no teeth lost before death, teeth very worn; 2 precondyles; large tympanic perforations
	ossicle above lambda; no teeth lost from upper jaw, teeth hardly worn; tympanic perforations R and L.
	above lambda; no teeth lost from either jaw, teeth hardly worn; JR.
	st before death; tympanic perforation R; J =; unusually feeble and small skull.
	worn, L 3rd M and R 2nd M lost from lower jaw before death; 2 marked precondyles.
	=.
	lly erupted; large epipteric bone L; slight tympanic perforations R and L; J =.
	; teeth complete and hardly worn; slight tympanic perforations R and L; JL.
	h; epipteric bones R and L; tympanic perforations R and L; JL; traces of sutures between ex- and supra-occipital bones
	d L; 3 or 4 teeth lost from both jaws before death, teeth greatly worn; marked tympanic perforations R and L; JR;
	xceedingly worn.
	L; tympanic perforation L; JL.
	h moderately worn; injury (? post mortem) at root of L mastoid; J =.
	re R and L; basal suture obliterated; no teeth lost before death, no 3rd M's in lower jaw, upper 3rd M's below alveolar
	pper jaw, teeth of lower jaw complete and greatly worn; J =.
	bone L; tympanic perforation L; JR.
	bliterated, sagittal and lambdoid sutures beginning to close; wormians in lambdoid suture R and L; no teeth lost before
	tal bone R.
	n lost before death (?); JR; fronto-temporal articulation R.
	M's not fully erupted in either jaw; JR.
	d suture R and L; teeth of both jaws complete but considerably worn; tympanic perforations R and L; JL.
	bably no teeth lost before death, 3rd M's hardly worn; defect of L tympanic plate (?); JR.
	pipteric bone R; JR; asymmetrical face.
	h complete and very worn; epipteric bone L; JL.
	th, no lower 3rd ML, teeth very worn; epipteric bones R and L; tympanic perforations R and L; JR.
	erated; wormians in R lambdoid suture; several teeth lost from both jaws; epipteric bones R and L; JR.
	h, no 3rd M's in both jaws, teeth exceedingly worn, one molar carious; JR.
	mplete (?); JR; prominent parietal bosses.
	death; tympanic perforation R; J =.
	w, teeth unworn; JR.

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Lengths with Callipers

Serial No.	Grave No.	Sex	w_1	w_2	h_1	zz	$c_r c_r$	rb'	G_2'	$c_y c_r$	$g_0 g_0$	$g_n g_0$ (L)	$g_n g_0$ (R)	$c_y l$	$c_y b$	m_1
2	No Mark	♂?	113·4	93·2	33·2	44·0	91·5	30·4	50·0	32·6	89·6	79·8	78·3	20·1	9·1	29
3	No Mark XI	♂	117·1	90·3	32·1	42·0	95·5	36·7	42·0	40·2	86·5	88·0	91·0	19·6	8·7	25
7	391 IX	♂	112·7	102·0	38·0	45·4	—	39·3	46·5	39·7 *R*	98·6	92·8	94·4	22·5	10·3	28
8	296 IX	♂	—	—	33·9	39·9	—	37·3	38·8	—	—	—	81·7	—	—	27
9	1591 IX	♂	119·8	97·2	35·6	44·0	91·3	34·4	44·4	33·7	95·6	87·7	86·7	24·0	10·8	31
10	1613 IX	♂?	—	91·2	—	45·4	89·2	31·6	46·8	36·8 *R*	86·2	79·9	80·3	19·8 *R*	8·5 *R*	28
11	1550 IX	♂	116·3	100·6	38·6?	44·9	101·3	33·8	49·5	37·2	97·7	93·7	91·5	21·0	11·1	30
13	1509 XI	♂?	—	—	35·5	—	—	—	45·8	—	—	—	81·4	—	—	27
14	1804 IX	♂	—	—	36·2	44·0	—	33·2 *R*	—	33·1 *R*	—	—	82·8	20·3 *R*	8·4 *R*	29
16	389 IX	♂	120·3	92·0?	29·8	45·5	87·9	35·9	43·0	36·3	88·0?	84·5	85·0?	20·1	8·2	24
17	1618 IX	♂?	110·7	91·4	32·0	47·0	87·4	30·0	45·3	34·0	88·4	88·3	87·8	19·2	9·5	27
19	1540 IX	♂?	—	—	—	—	—	—	—	—	—	—	—	—	—	
22	388 IX	♂?	—	87·2?	31·9	42·1	—	31·5	41·8	34·0	—	80·8	—	18·8	9·3	28
23	536 IX	♂	117·7	95·7	40·6	41·6	95·7	36·6 *R*	39·8	34·7	—	—	88·0	22·8	7·8	27
26	289 IX	♂?	117·5	89·5	31·7	42·7	89·9	30·7	48·0	31·6	86·1	78·6	80·6	23·1	9·7	29
27	1625 IX	♂	116·7	91·6	36·4	44·8	101·3	35·1	45·3	36·6	88·4	87·2	86·2	21·8	7·8	25
28	293 IX	♂	111·8	94·0	36·2	47·7	90·5	36·0	45·4	38·2	88·4	88·1	88·9	19·8	7·2	27
29	1603 IX	♂	—	92·1	29·4?	44·0	94·5	34·5	47·5	41·2 *R*	90·1	—	88·2	24·1 *R*	6·9 *R*	
31	1614 IX	♂	113·8	88·5	31·3	46·3	90·1	33·3	44·4	34·6	85·8	80·8	79·7	17·9	9·2	24
32	1653 IX	♂	116·7	101·8	35·8	46·0	99·0	34·4	48·8	33·5	98·3	93·2	91·4	21·2	9·7	29
35	1704 IX	♂?	—	92·8	30·3	45·4	—	30·2	47·7?	31·6	—	84·0	—	20·2?	10·2	27
38	1534 IX	♂	—	—	—	—	—	—	—	—	—	—	—	—	—	
39	1601*a* IX	♂	127·4	96·0	41·6	[illegible]	—	39·2	42·0	—	92·2	86·8	85·4	24·6	11·0	29
41	1542 IX	♂?	114·5	88·7	34·2	[illegible]	93·0	30·1	42·0	31·9	83·6	80·9	78·0	20·5	7·4	26
42	1549 IX	♂	108·4	95·0	31·4	42·4	—	38·0	45·0	33·5	89·6	87·0	86·4	19·2	10·0	27
44	1802 IX	♂	114·6	90·4	35·5	46·4	90·4	33·2	40·4	29·6	84·8	82·0	81·4	19·6	10·6	27
45	1600? IX	♂?	110·9	90·3	—	41·4	95·4	31·3	43·8	34·3	86·0	85·5	84·6	—	9·8	
46	1555 IX	♂	114·4	83·8	—	—	—	29·6	45·6?	39·5 *R*	79·6	82·6	81·6	21·3	8·4	
48	1529 IX	♂	—	—	—	—	—	—	—	—	—	—	—	—	—	
50	1562 IX	♂	—	86·4?	28·1	42·6	—	28·8	46·1	32·3	81·3	73·8	71·4	19·8	9·5	31
52	1727*a*? IX	♂	108·8	93·3	31·0	48·6	—	36·1	44·7	32·2	89·2	88·5	88·0	20·4	9·2	28
56	538 IX	♂	110·6	86·8	33·5	42·2	92·6	34·0	42·2	35·0	82·6	83·5	83·2	22·4	10·4	27
57	1728 IX	♂?	113·7	85·6	33·6	46·0	—	28·4	44·9	31·2 *R*	80·7	79·0	77·8	20·8	8·1	24
63	1527 IX	♂?	106·3?	84·2	32·1	42·8	80·5	34·0	43·7	38·7	80·4	76·0	75·5	18·8	8·2	26
64	1587 IX	♂?	105·7	89·4	33·2	42·6	82·3	33·2	40·9	28·6	85·2	81·6	82·3	18·7	8·6	27
65	1514 IX	♂	123·6	95·2	33·8?	41·7	97·3	30·6	—	38·4	91·1	83·5	81·4	22·3	7·4	
66	1593 IX	♂	—	—	—	—	—	—	—	—	—	—	—	—	—	
68	1579 IX	♂	108·6?	95·4	28·7	40·2	90·1	30·8	41·9	31·8	92·2	82·0	81·5	—	8·3	24
70	1700 IX	♂	—	—	32·6	43·7	—	31·4	44·2	—	—	84·2	—	—	8·0	26
71	1605 IX	♂?	—	93·2	33·1	42·2	—	32·6	49·6	34·4 *R*	89·6	82·5	83·2	—	6·3 *R*	28
1	1632 IX	♀	98·7	80·7	—	40·1	74·0	31·3	—	32·5	77·3	75·2	75·5	18·5	9·0	
4	No Mark IX	♀?	—	76·0	39·3	41·4	—	34·4 *R*	37·2	37·1 *R*	—	—	78·0	17·4 *R*	8·8 *R*	2
5	15?? IX	♀	—	—	—	—	—	—	—	—	—	—	—	—	—	
6	1586 IX	♀?	—	—	30·8	42·8	—	31·9 *R*	46·4	—	—	—	—	—	—	2
12	1526 IX	♀?	—	—	—	—	—	—	—	—	—	—	—	—	—	
15	1569 IX	♀?	120·6	98·3	28·8	44·4	92·3	33·3	—	33·0	93·9	79·1	80·5	22·0	9·0	2
18	1715 IX	♀?	101·6?	91·3	30·6	43·8	81·0	29·4	43·0	29·0	87·3	82·0	83·2	—	8·1	2
21	1601*d* IX	♀?	106·1	80·3	32·8	43·6	82·4	34·3	40·3	33·9	75·0	78·0	77·6	21·0 *R*	9·6	2
24	1540″ IX	♀	101·7	77·5	27·9	39·2	83·5	26·3	41·8	24·8	73·3	73·1	74·4	17·2	8·2	2
25	1506 IX	♀?	105·1	83·6	33·2	41·4	—	30·5	41·1	32·4	79·8	77·8	79·1	19·0	7·2	2
3[illegible]	541 IX	♀	105·9	84·0	29·9	44·6	83·4	32·3	45·8	30·4	76·7	75·8	76·9	19·3	9·6	2
33	1671 IX	♀?	—	—	29·0	44·0	—	31·4 *R*	40·7	30·6 *R*	—	—	74·6?	—	9·8 *R*	
34	1537 IX	♀?	119·0	84·0	32·0?	41·5	85·8	28·1	44·5	35·4	81·0	82·4	83·1	19·8	6·7	2
36	1558 IX	♀?	—	—	31·2?	—	—	31·4	42·8	30·7	—	78·5?	—	23·1	10·1	2
37	400 IX	♀?	113·0	95·0	30·8	41·5	88·1	28·1	43·7	30·0	90·3	79·5	79·6	17·2	7·8	2
40	1635	♀	101·0	78·7	26·8	39·5	74·5	30·9	40·5	32·7	75·2	74·1	76·0	17·4	7·8	2
43	387 IX	♀?	—	88·4	28·2	43·1	86·8	30·1	45·8	28·3	83·8	81·5	83·5	20·5	8·3	2
47	1504 IX	♀?	105·5	82·0	30·6	45·0	87·3	31·5	43·1	32·5	76·3	76·1	76·1	17·9 *R*	9·3	2
49	1803 IX	♀?	115·3	81·8	—	—	98·4	29·9	—	35·2	76·4	—	—	19·1	9·1	
51	1521 IX	♀?	—	—	—	—	—	—	—	—	—	—	—	—	—	
53	1541 IX	♀?	109·5	94·0	34·4	43·5	—	32·3	47·0	34·6 *R*	88·4	85·7	84·8	18·4 *R*	8·7 *R*	
54	1540‴ IX	♀?	116·0	95·9	29·8	40·7	91·0	26·5	48·6	28·4	91·0	79·2	78·3	18·6	7·6	2
55	+60*x* IX	♀	100·6	78·6	28·0	41·0	85·2	29·2	38·6	29·9	74·9	73·4	75·4	19·3	7·2	2
58	1538 IX	♀	100·5	84·8	30·3	42·0	80·0	29·0	43·6	31·4	82·1	76·4	79·6	19·1	8·4	2
59	1601*e* IX	♀	—	—	—	—	—	—	—	—	—	—	—	—	—	
60	1731 IX	♀	—	90·5	30·3	44·3	—	26·6	45·3	—	—	—	—	—	—	2
61	158[illegible] IX	♀?	—	—	—	—	—	—	—	33·2	—	—	—	17·3	8·1	
62	1672 IX	♀	114·2	90·8	34·0	46·4	89·6	31·5	44·[illegible]	30·8	84·0	82·4	81·6	20·7	8·6	2
67	1581 IX	♀?	105·6	82·3	31·0	40·7	80·5	30·3	44·2	30·4	74·8	79·2	80·2	17·6	7·5	2
69	1522 IX	♀?	—	—	—	—	—	—	—	—	—	—	—	—	—	
20	1601*b* IX	Juvenile sex?	99·2	81·9	29·7	43·5	81·7	32·5	40·0	30·0	76·6	74·7	76·4	16·7	9·3	2

Appendix II. *Individual Measurements of Sedment Mandibles.*

Tape							Lengths and Heights on Mandible Board											
$c_g l$	$c_g b$	$m_2 p_1$	$p_4 d_1$	$p_4 g_n$	$p_4 d_f$	$g_n d_f$	$g_0 p_4 g_0$	ih	ih'	$c_r h$	$c_g h$	$d_f h$	$\bullet_r h$	$p_1 h$	$c_g l$	rl	ml	100 $c_r h/ml$
20·1	9·1	29·2	29·8	6·6	26·2	30·4	177·0	49·8	12·1	67·0	58·2	35·0	23·9	28·0	70·0	62·6	96·1	69·7
19·6	8·7	25·6	28·3	6·0	24·5	28·8	199·0	46·4	16·2	65·8	59·8	36·9	26·4	30·6	83·0	62·4	105·9	62·1
22·5	10·3	28·5	32·0	9·8	27·8	34·4	215·0	59·0 *R*	13·2 *R*	77·0 *R*	75·1	40·3	31·9	34·4	86·1	80·8	107·3	71·8 *R*
—	—	27·2	26·6	9·8	23·1	29·8	—	52·8	—	72·5	—	35·3	26·9	29·8	76·8	—	97·7	74·2
24·0	10·8	31·6	27·1	9·0	27·1	33·8	193·0	49·0	13·6	71·1	58·4	38·8	28·4	35·3	78·0	63·2	106·3	66·9
19·8 *R*	8·5 *R*	28·3	—	10·9	28·9	26·6	183·5	54·1 *R*	12·8 *R*	68·6	62·0 *R*	—	26·2	31·0	73·8	66·0 *R*	101·7	67·5
21·0	11·1	30·0	32·0	12·2	28·3	35·4	212·0	55·5	14·8	77·6	66·0	40·4	—	36·3	83·0	71·0	103·3	75·1
—	—	27·6	27·2	10·8	23·7	32·3	—	—	—	—	55·0 *R*	36·4	27·0	30·3	76·3	59·3 *R*	107·5	—
20·3 *R*	8·4 *R*	29·8	32·7	8·4	29·0	34·0	—	50·2 *R*	14·0 *R*	72·5 *R*	56·8 *R*	37·9	27·9	34·7	79·0	63·6 *R*	110·1	65·8 *R*
20·1	8·2	24·8	24·8	8·3	21·3	27·6	189·5?	43·7	13·2	60·0	54·5	35·0	26·2	32·9	77·3	59·0	101·7	59·0
19·2	9·5	27·5	25·0	9·8	22·5	29·3	178·5	43·3	13·9	65·2	50·6	36·0	25·8	31·7	69·8	57·8	99·6	65·5
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
18·8	9·3	28·2	23·6	6·2	23·7	27·2	—	42·2	13·0	65·0	48·7	35·3	22·8	25·9	73·0	55·5	101·3	64·2
22·8	7·8	27·2	33·9	9·1	30·7	38·0	—	60·3	12·2	78·1	69·0	41·7	32·0	36·8	78·0	75·0	108·6	71·9
23·1	9·7	29·5	28·4	5·3	25·1	27·8	183·0	50·6	11·9	67·6	59·0	36·2	25·8	28·8	70·6	64·9	96·8	69·8
21·8	7·8	25·3	31·0	8·2	27·8	33·3	196·0	44·0	14·0	69·0	50·0	41·8	28·6	35·5	78·5	57·0	101·0	68·3
19·8	7·2	27·8	27·8	11·4	23·8	32·8	207·0	47·0	14·3	67·0	59·0	40·2	26·4	33·8	84·5	64·0	102·0	65·7
24·1 *R*	6·9 *R*	—	25·2?	9·0	—	—	199·0?	44·7	16·3	65·9	57·9 *R*	—	26·7	32·1	79·0	65·6 *R*	100·0	65·9
17·9	9·2	24·5	27·3	9·4	24·8	29·8	187·5	47·1	13·0	66·5	58·1	34·8	27·2	32·2	73·2	62·0	102·1	65·1
21·2	9·7	29·0	30·7	8·6	25·8	31·1	210·0	54·9	13·4	72·3	66·6	41·8	28·8	34·8	84·6	69·3	110·0	65·7
20·2?	10·2	27·5	25·0	7·1	21·0	26·4	—	50·4	13·2	69·7	58·5	35·4	25·8	30·2	72·8	62·8	97·0	71·9
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
24·6	11·0	29·5	35·2	8·7	32·7	38·8	196·0	54·8	—	—	69·0	43·8	32·7	38·2	78·4	73·7	109·0	—
20·5	7·4	26·4	31·1	5·7	27·2	30·8	182·0	41·0	13·3	68·1	48·1	29·2	22·1	34·9	69·5	54·0	98·0	69·5
19·2	10·0	27·4	26·3	8·8	22·3	28·0	202·5	41·8	13·2 *R*	64·2 *R*	47·7	36·2	24·3	28·6	78·9	56·5	108·4	59·2 *R*
19·6	10·6	27·2	31·2	7·3	28·5	33·8	189·0	43·0	13·8	63·0	50·7	38·0	24·0	30·7	73·5	56·6	101·4	62·1
—	9·8	—	—	7·4	—	—	192·5	43·2	11·0	58·3	52·0	—	25·2	—	77·0	56·0	104·0	56·1
21·3	8·4	—	—	5·5	—	—	187·0	42·7	11·2 *R*	59·0 *R*	51·0	—	27·9	—	74·0	57·0	102·5	57·6 *R*
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
19·8	9·5	31·0	23·6	8·3	20·7	25·8	187·0	51·0	14·1	68·0	62·8	33·2	27·8	22·6	65·0	68·0	90·8	74·9
20·4	9·2	28·4	25·2	8·1	22·8	27·8	201·0	54·2	14·1	71·8	65·1	36·9	27·2	33·1	80·0	69·9	109·1	65·8
22·4	10·4	27·8	27·8	8·3	23·7	29·3	188·0	56·3	13·2	77·4	68·0	34·3	31·2	30·2	77·5	69·6	97·7	79·2
20·8	8·1	24·4	28·7	8·6	25·5	31·0	180·0	46·8	10·0 *R*	61·8 *R*	53·0	37·1	23·9	31·6	71·2	60·9	104·8	59·0 *R*
18·8	8·2	26·1	26·4	8·8	23·3	29·8	176·0	41·7	14·0	60·9	54·8	36·7	27·2	31·1	69·7	57·3	95·0	64·1
18·7	8·6	27·8	28·1	8·2	24·8	30·2	190·0	44·1	13·6	65·9	52·9	35·9	29·8	25·2	74·8	55·7	95·8	68·8
22·3	7·4	—	—	7·3	—	—	184·5	45·8	14·0	65·8	54·4	35·3?	—	—	72·2	65·3	108·8	60·5
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	8·3	24·5	26·2	6·2	23·2	27·3	182·5	43·5	12·4	61·1	53·0	32·1	24·0	28·3	72·0	57·0	97·3	62·8
—	8·0	29·6	28·4	6·0	25·3	29·1	—	44·6	—	—	52·4	37·2	27·2	31·6	73·2	60·0	110·2	—
—	6·3 *R*	28·4	28·7	6·5	25·4	30·3	189·5	50·3 *R*	12·3 *R*	72·0 *R*	57·3 *R*	37·0	24·8	28·4	73·6	64·3 *R*	103·0	69·9 *R*
18·5	9·0	—	—	6·8	—	—	168·0	48·3	11·9	60·4	61·0	—	24·2	—	70·1	63·0	93·2	64·8
17·4 *R*	8·8 *R*	27·0	25·2	6·7	22·0	26·3	—	42·0 *R*	11·8 *R*	60·3 *R*	50·0 *R*	33·6	25·2	28·3	76·0?	54·8 *R*	102·8	58·7 *R*
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	28·1	25·5	7·3	22·3	28·2	—	41·6 *R*	—	56·6 *R*	—	34·9	21·9	28·1	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
22·0	9·0	26·5	32·8	8·1	19·6	26·0	178·0	39·6	9·6	56·9	41·6	30·9	25·3	27·3	69·0	53·0	101·4	56·1
—	8·1	27·5	30·2	6·7	23·2	27·0	184·5	43·8	14·8	64·0	53·9	34·2	23·8	29·9	73·8	59·6	99·7	64·2
21·0 *R*	9·6	26·7	26·5	8·1	23·6	30·2	178·5	47·1	13·4	64·0	58·2	33·3	26·8	29·2	73·1	61·5	96·2	66·5
17·2	8·2	28·2	24·3	6·0	25·8	24·3	169·0	42·9	11·2	60·3	50·0	35·6	22·8	28·0	68·3	54·8	88·4	68·2
19·0	7·2	28·2	29·4	8·1	23·3	27·8	179·5	41·5	14·6	62·6	48·0	38·8	23·9	32·3	72·2	58·0	106·0	59·1
19·3	9·6	27·0	24·0	8·2	20·2	26·3	173·5	41·4	10·9	56·0	48·0	32·3	23·4	27·0	69·7	52·2	95·2	58·8
—	9·8 *R*	—	21·8	9·0	19·9	27·4	—	44·9 *R*	13·2 *R*	63·0 *R*	55·4 *R*	32·4	25·7	26·8	70·2	57·5	94·8	66·5 *R*
19·8	6·7	27·1	26·8?	6·0	25·8	30·0	183·0	43·5	8·6	57·4	52·0	35·0?	22·0	29·8	74·8	55·7	101·6	56·5
23·1	10·1	25·0	27·6?	6·5	24·7	29·3	—	45·4	12·1	63·2	53·0	35·8?	26·9	30·8	72·2?	59·0?	96·6	65·4
17·2	7·8	26·3	28·0	6·3	24·0	27·2	177·0	41·8	14·9	60·0	52·2	34·3	23·8	28·4	69·1	60·0	101·6	59·1
17·4	7·8	27·8	23·0	6·8	19·0	23·8	169·0	41·6	12·3	57·0	52·0	30·7	21·8	25·1	69·0	56·8	91·3	62·4
20·5	8·3	25·4	25·0	6·8	23·1	26·6	187·5	49·6	10·9	65·9	57·2	32·9	26·4	28·4	76·8	60·0	93·2	70·7
17·9 *R*	9·3	26·4	23·3	9·7	19·8	26·2	179·5	45·7	12·8	62·0	56·0	32·8	28·0	26·0	70·8	61·2	100·8	61·5
19·1	9·1	—	—	—	—	—	—	44·5	13·1	64·0	53·2	—	—	—	—	63·0	—	—
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
18·4 *R*	8·7 *R*	—	34·1	7·8	25·2	30·6	192·0	41·8	13·8 *R*	65·0 *R*	55·1	35·9	26·2	—	79·0	58·4	102·2	63·6 *R*
18·6	7·6	28·2	27·7	5·4	24·0	26·2	177·0	37·2	11·6	58·1	41·0	34·2	18·9	26·3	66·9	53·8	105·2	55·2
19·5	7·2	26·6?	23·8	5·8	21·8	25·8	170·0	39·4	10·7	57·4	46·0	33·3	20·9	27·2?	67·2	52·1	92·0	62·4
19·1	8·4	27·3	27·0	5·9	24·0	27·6	173·0	42·4	14·8	68·0	51·5	33·7	21·8	30·1	69·0	58·0	99·0	68·7
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
—	—	26·7	23·8	5·9	23·7	27·2	—	43·0 *R*	—	62·2 *R*	—	33·8	22·4	30·0	—	—	—	—
17·3	8·1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
20·7	8·6	27·9	28·3	7·8	25·6	30·1	186·5	39·4	11·3	57·0	46·0	39·0	28·7	33·8	73·0	52·0	101·3	56·3
17·6	7·5	26·8?	26·8	6·3	24·7?	28·8?	183·0	36·8	9·0	51·0	45·1	35·7	23·2	33·1?	73·1	48·5	97·0	52·6
—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
16·7	9·3	28·4	25·8	6·8	22·4	26·5	174·5	41·4	11·2	56·7	49·4	31·3	23·8	26·0	69·5	52·3	90·0	63·0

ividual Measurements of Sedment Mandibles.

	Lengths and Heights on Mandible Board									Indices								
	ih′	c_1h	c_2h	d_1h	d_2h	p_1h	c_pl	*rl*	*ml*	100 c_ph/ml	100 c_rc_r/ml	100 g_1g_2/c_pl	100 *rb′/rl*	100 c_pb/c_pl	100 g_1g_2/c_rc_r	100 c_ph/c_1h	100 *ih′/c₂c_r*	100 d_1h/c_r
	12·1	67·0	58·2	35·0	23·9	28·0	70·0	62·6	96·1	69·7	95·2	128·0	48·6	45·3	97·9	86·9	37·1	52·2
	16·2	65·8	59·8	36·9	26·4	30·6	83·0	62·4	105·9	62·1	90·2	104·2	58·8	44·4	90·6	90·9	40·3	56·1
R	13·2 *R*	77·0 *R*	75·1	40·3	31·9	34·4	86·1	80·8	107·3	71·8 *R*	—	114·5	48·6	45·8	—	97·5	33·2 *R*	52·3
	—	72·5	—	35·3	26·9	29·8	76·8	—	97·7	74·2	—	—	—	—	—	—	—	48·7
	13·6	71·1	58·4	38·8	28·4	35·3	78·0	63·2	106·3	66·9	85·9	122·6	54·4	45·0	104·7	82·1	40·4	54·6
R	12·8 *R*	68·6	62·0 *R*	—	26·2	31·0	73·8	66·0 *R*	101·7	67·5	87·7	116·8	47·9	42·9 *R*	96·6	90·4	34·8 *R*	—
	14·8	77·6	66·0	40·4	—	36·3	83·0	71·0	103·3	75·1	98·1	117·7	47·6	52·9	96·4	85·1	39·8	52·1
	—	—	55·0 *R*	36·4	27·0	30·3	76·3	59·3 *R*	107·5	—	—	—	—	—	—	—	—	—
R	14·0 *R*	72·5 *R*	56·8 *R*	37·9	27·9	34·7	79·0	63·6 *R*	110·1	65·8 *R*	—	—	52·2 *R*	41·4 *R*	—	78·3 *R*	42·3 *R*	52·3
	13·2	60·0	54·5	35·0	26·2	32·9	77·3	59·0	101·7	59·0	86·4	113·8?	60·8	40·8	100·1?	90·8	36·4	58·3
	13·9	65·2	50·6	36·0	25·8	31·7	69·8	57·8	99·6	65·5	87·8	126·6	51·9	49·5	101·1	77·6	40·9	55·2
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	13·0	65·0	48·7	35·3	22·8	25·9	73·0	55·5	101·3	64·2	—	—	56·8	49·5	—	74·9	38·2	54·3
	12·2	78·1	69·0	41·7	32·0	36·8	78·0	75·0	108·6	71·9	88·1	—	48·8	34·2	—	88·3	35·2	53·4
	11·9	67·6	59·0	36·2	25·8	28·8	70·6	64·9	96·8	69·8	92·9	122·0	47·3	42·0	95·8	87·3	37·7	53·6
	14·0	69·0	50·0	41·8	28·6	35·5	78·5	57·0	101·0	68·3	100·3	112·6	61·6	35·8	87·3	72·5	38·3	60·6
	14·3	67·0	59·0	40·2	26·4	33·8	84·5	64·0	102·0	65·7	88·7	104·6	56·3	36·4	97·7	88·1	37·4	60·0
	16·3	65·9	57·9 *R*	—	26·7	32·1	79·0	65·6 *R*	100·0	65·9	94·5	114·1	52·6	28·6 *R*	95·3	87·9	39·6	—
	13·0	66·5	58·1	34·8	27·2	32·2	73·2	62·0	102·1	65·1	88·2	117·2	53·7	51·4	95·2	87·4	37·6	52·3
	13·4	72·3	66·6	41·8	28·8	34·8	84·6	69·3	110·0	65·7	90·0	116·2	49·6	45·8	99·3	92·1	40·0	57·8
	13·2	69·7	58·5	35·4	25·8	30·2	72·8	62·8	97·0	71·9	—	—	48·1	50·5?	—	83·9	41·8	50·8
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	69·0	43·8	32·7	38·2	78·4	73·7	109·0	—	—	117·6	53·2	44·7	—	—	—	—
	13·3	68·1	48·1	29·2	22·1	34·9	69·5	54·0	98·0	69·5	94·9	120·3	55·7	36·1	89·9	70·6	41·7	42·9
	13·2 *R*	64·2 *R*	49·7	36·2	24·3	28·6	78·9	56·5	108·4	59·2 *R*	—	113·6	67·3	52·1	—	77·4	39·4	56·4
	13·8	63·0	50·7	38·0	24·0	30·7	73·5	56·6	101·4	62·1	89·2	115·4	58·7	54·1	93·8	80·5	48·6	60·3
	11·0	58·3	52·0	—	25·2	—	77·0	56·0	104·0	56·1	91·7	111·7	55·9	—	90·1	89·2	32·1	—
	11·2 *R*	59·0 *R*	51·0	—	27·9	—	74·0	57·0	102·5	57·6 *R*	—	107·6	51·9	39·4	—	86·4	28·5 *R*	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	14·1	68·0	62·8	33·2	27·8	22·6	65·0	68·0	90·8	74·9	—	125·1	42·4	48·0	—	92·4	43·7	48·8
	14·1	71·8	65·1	36·9	27·2	33·1	80·0	69·9	109·1	65·8	—	111·5	51·6	45·1	—	90·7	43·8	51·4
	13·2	77·4	68·0	34·3	31·2	30·2	77·5	69·6	97·7	79·2	94·8	106·6	48·9	46·4	89·2	87·9	37·7	44·3
	10·0 *R*	61·8 *R*	53·0	37·1	23·9	31·6	71·2	60·9	104·8	59·0 *R*	—	113·3	46·6	38·9	—	85·8	32·1 *R*	60·0
	14·0	60·9	54·8	36·7	27·2	31·1	69·7	57·3	95·0	64·1	84·7	115·4	59·3	43·6	99·9	90·0	36·2	60·3
	13·6	65·9	52·9	35·9	29·8	25·2	74·8	55·7	95·8	68·8	85·9	113·9	59·6	46·0	103·1	80·3	47·6	54·5
	14·0	65·8	54·4	35·3?	—	—	72·2	65·3	108·8	60·5	89·4	126·2	46·9	33·2	93·6	82·7	36·5	53·6
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	12·4	61·1	53·0	32·1	24·0	28·3	72·0	57·0	97·3	62·8	92·6	128·1	54·0	—	102·3	86·7	39·0	52·5
	—	—	52·4	37·2	27·2	31·6	73·2	60·0	110·2	—	—	—	52·3	—	—	—	—	—
R	12·3 *R*	72·0 *R*	57·3 *R*	37·0	24·8	28·4	73·6	64·3 *R*	103·0	69·9 *R*	—	121·7	50·7	—	—	79·6 *R*	35·8 *R*	51·4
	11·9	60·4	61·0	—	24·2	—	70·1	63·0	93·2	64·8	79·4	110·3	49·7	48·6	104·5	101·0	36·6	—
R	11·8 *R*	60·3 *R*	50·0 *R*	33·6	25·2	28·3	76·0?	54·8 *R*	102·8	58·7 *R*	—	—	62·8 *R*	50·6 *R*	—	82·9 *R*	31·8 *R*	55·7
R	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	56·6 *R*	—	34·9	21·9	28·1	—	—	—	—	—	—	—	—	—	—	—	61·7
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	9·6	56·9	41·6	30·9	25·3	27·3	69·0	53·0	101·4	56·1	91·0	136·1	62·8	40·9	101·7	73·1	29·1	54·3
	14·8	64·0	53·9	34·2	23·8	29·9	73·8	59·6	99·7	64·2	81·2	118·3	49·3	—	107·8	84·2	51·0	53·4
	13·4	64·0	58·2	33·3	26·8	29·2	73·1	61·5	96·2	66·5	85·7	102·6	55·8	45·7	91·0	90·9	39·5	52·0
	11·2	60·3	50·0	35·6	22·8	28·0	68·3	54·8	88·4	68·2	94·5	107·3	48·0	47·7	87·8	82·9	45·2	59·0
	14·6	62·6	48·0	38·8	23·9	32·3	72·2	58·0	106·0	59·1	—	110·5	52·6	37·9	—	76·7	45·1	62·0
	10·9	56·0	48·0	32·3	23·4	27·0	69·7	52·2	95·2	58·8	87·6	110·0	61·9	49·7	92·0	85·7	35·9	57·7
?	13·2 *R*	63·0 *R*	55·4 *R*	32·4	25·7	26·8	70·2	57·5	94·8	66·5 *R*	—	—	54·6	—	—	87·9 *R*	43·1 *R*	51·4
	8·6	57·4	52·0	35·0?	22·0	29·8	74·8	55·7	101·6	56·5	84·4	108·3	50·4	33·8	94·4	90·6	24·3	61·0
	12·1	63·2	53·0	35·8?	26·9	30·8	72·2?	59·0?	96·6	65·4	—	—	53·2?	43·7	—	83·9	39·4	56·6
	14·9	60·0	52·2	34·3	23·8	28·4	69·1	60·0	101·6	59·1	86·7	130·7	46·8	45·3	102·5	87·0	49·7	57·2
	12·3	57·0	52·0	30·7	21·8	25·1	69·0	56·8	91·3	62·4	81·6	109·0	54·4	44·8	100·9	91·2	37·6	53·9
	10·9	65·9	57·2	32·9	26·4	28·4	76·8	60·0	93·2	70·7	93·1	109·1	50·2	40·5	96·5	86·8	38·5	49·9
	12·8	62·0	56·0	32·8	28·0	26·0	70·8	61·2	100·8	61·5	86·6	107·8	51·5	52·0	87·4	90·3	39·4	52·9
	13·1	64·0	53·2	—	—	—	—	63·0	—	—	—	—	47·5	47·6	77·6	83·1	37·2	—
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	13·8 *R*	65·0 *R*	55·1	35·9	26·2	—	79·0	58·4	102·2	63·6 *R*	—	111·9	55·3	47·3 *R*	—	84·8	39·9 *R*	55·2
	11·6	58·1	41·0	34·2	18·9	26·3	66·9	53·8	105·2	55·2	86·5	136·0	49·3	40·9	100·0	70·6	40·8	58·9
	10·7	57·4	46·0	33·3	20·9	27·2?	67·2	52·1	92·0	62·4	92·6	111·5	56·0	36·9	87·9	80·1	35·8	58·0
	14·8	68·0	51·5	33·7	21·8	30·1	69·0	58·0	99·0	68·7	80·8	119·0	50·0	44·0	102·6	75·7	47·1	49·6
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	—	62·2 *R*	—	33·8	22·4	30·0	—	—	—	—	—	—	—	—	—	—	—	—
	—	—	—	—	—	—	—	56·1	—	—	—	—	—	46·8	—	—	—	—
	11·3	57·0	46·0	39·0	28·7	33·8	73·0	52·0	101·3	56·3	88·5	115·1	60·6	41·3	93·8	80·7	36·7	68·4
	9·0	51·0	45·1	35·7	23·2	33·1?	73·1	48·5	97·0	52·6	83·0	102·3	62·5	42·6	92·9	88·4	29·6	70·0
	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—
	11·2	56·7	49·4	31·3	23·8	26·0	69·5	52·3	90·0	63·0	90·8	110·2	62·1	55·7	93·8	87·1	37·3	55·2

Angles in degrees

100 $ih'/c_y c_r$	100 $d_h h/c_r h$	$M\angle$	$R\angle$	$G\angle$	$C\angle$	$C'\angle$	$L\angle$	$L'\angle$	$F\angle$	$S\angle$
37·1	52·2	117·0	77·5	68·9	74·7	81·5	—	—	—	—
40·3	56·1	115·5	75·0	57·5	71·0	76·0	98·5	95·0	92·0	22·5
33·2 *R*	52·3 *R*	116·5	73·0	63·2	67·1	72·1	81·5	79·0	86·0	9·4
—	48·7	113·5?	82·5?	—	77·0	78·9	93·2	91·3	89·0	14·3
40·4	54·6	121·0	78·0	66·3	78·5	79·1	92·3	92·6	81·3	13·2
34·8 *R*	—	118·5 *R*	72·1 *R*	64·9	—	74·8	90·5	88·6	94·2	15·7
39·8	52·1	118·5	79·0	63·5	63·8	64·0	—	—	—	—
—	—	121·0 *R*	82·0 *R*	—	80·4	82·6	107·4	107·0	92·0	24·8
42·3 *R*	52·3 *R*	123·0 *R*	87·0 *R*	—	64·8	66·0	—	—	—	—
36·4	58·3	118·5	69·5	62·4?	63·2	66·0	86·0	85·5	93·0	20·0
40·9	55·2	125·5	82·3	60·1	79·0	81·0	101·3	101·8	92·1	20·3
—	—	—	—	—	—	—	—	—	—	—
38·2	54·3	124·5	85·0	—	70·4	73·2	98·5	94·5	92·4	25·3
35·2	53·4	119·5	75·5	—	82·7	86·0	108·5	109·8	92·6	22·5
37·7	53·6	119·0	76·5	65·2	65·7	65·2	82·3	81·0	89·2	17·1
38·3	60·6	126·0	84·0	61·2	78·0	79·2	109·2	104·0	91·2	30·0
37·4	60·0	120·0	73·0	60·0	77·0	77·2	94·2	94·6	88·2	17·0
39·6	—	122·0 *R*	68·0 *R*	—	—	—	—	—	—	—
37·6	52·3	119·5	74·0	64·3	72·7	73·8	—	—	—	—
40·0	57·8	115·5	74·0	64·2	70·8	71·8	82·0	80·6	89·4	10·2
41·8	50·8	118·5	75·5	—	72·7	74·0	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—
—	—	118·5	—	64·6	81·0	81·4	99·0	98·0	92·0	17·6
41·7	42·9	124·5	98·5	63·2	72·0	74·9	93·5	93·3	92·9	18·6
39·4	56·4 *R*	126·0	73·0	62·0	73·4	76·8	93·1	93·0	87·5	16·3
48·6	60·3	125·0	79·0	62·3	76·1	78·0	102·0	101·6	91·8	24·0
32·1	—	124·0	67·5	60·5	—	—	—	—	—	—
28·5 *R*	—	125·5	76·0	58·0	—	—	97·6	97·8	93·2	—
—	—	—	—	—	—	—	—	—	—	—
43·7	48·8	120·5	75·0	68·0	69·0	66·4	—	—	—	—
43·8	51·4	118·0	72·5	60·6	75·3	75·8	84·0	81·8	81·0	8·2
37·7	44·3	112·5	86·6	59·0	78·0	80·7	90·2	90·0	89·2	9·5
32·1 *R*	60·0 *R*	128·5	66·5	61·8	68·9	69·6	83·0	81·0	86·5	13·4
36·2	60·3	121·5	71·5	63·9	83·8	84·0	—	—	—	—
47·6	54·5	115·5	88·0	62·5	81·0	81·6	98·0	99·3	92·8	16·4
36·5	53·6?	129·5	69·5	67·0	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—
39·0	52·5	122·1	70·5	68·6	71·0	71·3	84·8	83·3	89·8	13·5
—	—	129·5	—	—	73·0	72·3	—	—	—	—
35·8 *R*	51·4 *R*	122·5 *R*	83·0 *R*	65·3	80·1	81·6	103·0	102·7	95·1	21·4
36·6	—	117·5	65·0	61·8	—	—	—	—	—	—
31·8 *R*	55·7 *R*	121·5 *R*	77·5 *R*	—	78·3	80·2	103·6	101·8	95·0	23·4
—	—	—	—	—	—	—	—	—	—	—
—	61·7 *R*	—	—	—	82·0	83·0	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—
29·1	54·3	134·0	73·5	71·7	74·5	80·0	107·8	109·6	97·3	27·8
51·0	53·4	122·0	79·5	63·7	64·8	65·8	85·2	81·3	85·6	19·4
39·5	52·0	114·5	76·0	57·4	76·8	79·5	97·0	97·6	94·3	17·5
45·2	59·0	115·0	81·5	59·5	73·0	72·1	92·0	90·0	90·1	19·9
45·1	62·0	128·0	74·0	61·0	66·0	63·4	94·0	87·3	88·1	30·6
35·9	57·7	123·0	73·0	60·1	75·0	75·7	97·6	96·5	92·1	21·9
43·1 *R*	51·4 *R*	116·0 *R*	78·0 *R*	—	87·0	89·0	105·2	105·0	92·9	16·2
24·3	61·0?	124·5	69·5	58·6	70·7	68·7	—	—	—	—
39·4	56·6?	125·5?	73·0?	—	64·8	68·0	94·0	92·1	91·2	26·0
49·7	57·2	126·5	68·5	69·1	59·0	61·0	85·8	84·0	89·0	24·8
37·6	53·9	117·5	70·0	59·9	69·1	70·1	90·0	88·6	89·6	19·9
38·5	49·9	113·5	82·0	60·8	58·9	58·3	80·8	74·4	77·8	22·5
39·4	52·9	125·0	66·0	59·9	74·0	72·0	93·9	94·0	92·0	21·9
37·2	—	125·5	70·5	—	—	—	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—
39·9 *R*	55·2 *R*	120·0	75·0	71·9	67·3	70·2	103·4	103·4	95·6	33·2
40·8	58·9	137·5	78·0	70·2	54·4	55·6	81·0	76·5	87·0	25·4
35·8	58·0	121·0	80·5	60·0	81·6	83·0	—	—	—	—
47·1	49·6	125·5	85·5	63·3	64·8	67·0	88·4	87·0	88·9	21·4
—	—	—	—	—	—	—	—	—	—	—
—	—	—	—	—	68·5	68·0	—	—	—	—
—	—	131·0	77·5	—	—	—	—	—	—	—
36·7	68·4	128·5	73·1	61·3	66·0	64·0	97·8	100·3	92·0	33·8
29·6	70·0	125·0	68·0	55·8	72·7?	74·0?	—	—	—	—
—	—	—	—	—	—	—	—	—	—	—
37·3	55·2	118·0	77·0	60·9	71·8	71·9	91·0	89·8	91·2	19·1

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with the slightly differing early Egyptian types by our present series. These relationships suggest that we are dealing with a sample from a population which was predominantly of Egyptian origin and they may be taken to indicate that at some unknown period there was a direct, or indirect, link between the native Egyptians of Sedment and the Cretan people. The evidence is not sufficient to warrant any more definite statement. The Sedment has a higher cephalic index, a greater height and a higher height-length index than any other early Egyptian type: it is also differentiated by a high occipital index indicating that the arc from lambda to opisthion is less convex than usual. For the first three of these characters the divergences are in the direction of the modern Cretan type, but data for that type are not available in the case of the occipital index. Male and female type contours and mandibular measurements are presented, but owing to the lack of sufficient comparative material no definite conclusions can be deduced from these. The maximum breadth of the transverse section is relatively higher for the Sedment than for any other series for which transverse type contours have been published.

In conclusion I should like to express my great gratitude to Professor Karl Pearson for permitting me to undertake this research in his Laboratory, to Dr Morant for aid in many ways and to Miss McLearn for drawing the type contours.

A STATISTICAL STUDY OF CERTAIN ANTHROPOMETRIC MEASUREMENTS FROM SWEDEN.

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1. *Introduction.* The present paper consists of a statistical study of certain anthropometric data from Sweden, which have been taken from *The Racial Characters of the Swedish Nation*, edited by H. Lundborg and F. J. Linders, and published by the Swedish State Institute for Race Biology, Uppsala, in 1926*. My aim is to make a first application of the Coefficient of Racial Likeness to the discrimination of racial differences to be ascertained from measurements on the living. Hitherto the method of the C. R. L. has been applied chiefly to craniometric data†. As I have indicated in an earlier memoir (*Biometrika*, Vol. xx[A]. pp. 1—31) the want of standardisation renders analysis of measurements on the living by this, or indeed by any other, method largely futile.

The material consists of measurements of 46,983 conscripts and regular soldiers belonging to the Swedish Army and Navy. The subjects were all born in Sweden and were over 20 and under 22 years of age. The measurements were taken in 1922 and 1923, each person being measured by two observers. Special precautions were taken to ensure the same standards being maintained by all observers. One of the examiners measured the entire naval force, and another examined nearly half of the persons included in the investigation. The total number of observers was small, and all of them were connected with the Swedish State Institute for Race Biology. It may be assumed therefore that the present series of measurements are standardised and comparable *inter se*. Measurements of 404 persons born in foreign countries are available, but they were excluded from my analysis.

The birthplace of the person examined was chosen as the basis for the regional grouping of the material into five territories:

(A) *North Sweden*, comprising the provinces of Lappland, Västerbotten and Ångermanland.

(B) *West Sweden*, comprising Jämtland, Härjedalen, Dalarne, Värmland, Västmanland, Närke, Dalsland, Bohuslän and Västergötland.

(C) *East Sweden*, comprising Medelpad, Hälsingland, Gästrikland, Uppland, Södermanland, Östergötland, Småland and Öland Island.

(D) *South Sweden*, comprising Halland, Skåne, Blekinge and Gotland Island.

(E) The four biggest *Cities*: Stockholm, Göteborg, Malmö and Norrköping.

* I am much indebted to Professor H. Lundborg for kindly sending me a copy of this book immediately after its publication.

† In *Biometrika* it has been applied to racial characters in silkworms and to those of Macedonian local groups.

The material from each territory was further classified into four groups on an occupational basis:

(α) *Agricultural* communities in which, according to the 1910 Census, more than 60 % of the inhabitants earned their livelihood through agriculture, forestry and fishing.

(β) *Mixed* communities in which the corresponding percentage was under 60 but over 30.

(γ) *Industrial* communities in which the percentage was less than 30.

(δ) A fourth group, the *Urban* communities, consisted of the inhabitants of the cities, towns and market towns (exclusive of (E)).

We thus get the scheme (shown in Table I) for the whole of Sweden divided into 17 sections.

TABLE I.

Divisions of the Population of Sweden.

Territory	Occupational Group	Section Number	Number of persons examined
(A) North	Agricultural	1	2993
	Mixed ...	2	1059
	Industrial	3	406
	Urban ...	4	337
(B) West	Agricultural	5	7054
	Mixed ...	6	3200
	Industrial	7	1245
	Urban ...	8	1723
(C) East	Agricultural	9	6496
	Mixed ...	10	4642
	Industrial	11	1894
	Urban ...	12	2465
(D) South	Agricultural	13	3687
	Mixed ...	14	2665
	Industrial	15	625
	Urban ...	16	1737
(E) Four Largest Cities		17	4755
		Total	46,983

Mean values of the different characters for each section are shown in Table III on the following page.

Pooling certain of the above sections we obtain the geographical territories and occupational classes shown in Table II.

Several of the mean values for the occupational classes given in Table III were calculated by me; the other figures were taken from the published volume.

I may note here that after a careful comparison with the Census figures for the whole of Sweden the authors came to the conclusion that the geographical as well

as the occupational and social distributions of the persons measured were representative of the whole population. In other words, the present material may be considered to be a fair sample of the male Swedish population for the age-group 20—22 years*.

The values of the general means and standard deviations for the total sample are given in Table IV. The standard deviations for Bi-acromial Index, Supra-

TABLE II.

Divisions of the Population of Sweden (continued).

Territory	Number examined	Occupational Class	Number examined
(A) North	4,795	(α) Agricultural	20,230
(B) West	13,222	(β) Mixed ...	11,566
(C) East	15,497	(γ) Industrial	4,170
(D) South	8,714	(δ) Urban ...	6,262
(E) Cities	4,755	(E) Cities ...	4,755
Total	46,983	Total	46,983

TABLE IV.

General Means, Standard Deviations and Coefficients of Variation, with their Probable Errors†, based on the total Population of 46,983.

(Body measurements are in cms. and head measurements in mms.)

Character	Mean	Standard Deviation	Coefficient of Variation
1. Stature	172·23 ± ·018	5·93 ± ·013	3·44 ± ·008
2. Supra-sternal Height ...	140·89 ± ·013	5·29 ± ·012	3·75 ± ·008
3. Trunk Length	52·37 ± ·007	2·41 ± ·005	4·60 ± ·010
4. Arm Length	78·46 ± ·010	3·34 ± ·007	4·26 ± ·009
5. Leg Length	92·02 ± ·013	4·30 ± ·009	4·67 ± ·010
6. Bi-acromial Diameter ...	39·23 ± ·005	1·67 ± ·004	4·26 ± ·009
7. Inter-iliocristal Breadth ...	28·80 ± ·005	1·52 ± ·003	5·27 ± ·012
8. Trunk Length Index ...	30·49 ± ·004	1·18 ± ·003	3·88 ± ·009
9. Leg Length Index	53·43 ± ·004	1·29 ± ·003	2·41 ± ·005
10. Bi-acromial Index	22·80 ± ·003	0·92 ± ·002	4·04 ± ·009
11. Head Length	193·84 ± ·019	6·19 ± ·014	3·20 ± ·007
12. Head Breadth	150·44 ± ·016	5·10 ± ·011	3·39 ± ·007
13. Face Breadth	136·02 ± ·015	4·84 ± ·011	3·56 ± ·008
14. Morphological Face Height	126·57 ± ·022	6·92 ± ·015	5·46 ± ·012
15. Minimum Frontal Diameter	104·57 ± ·013	4·33 ± ·010	4·14 ± ·009
16. Cephalic Index	77·69 ± ·010	3·14 ± ·007	4·04 ± ·009
17. Morphological Face Index...	93·14 ± ·017	5·61 ± ·012	6·02 ± ·013

* Lundborg and Linders say: "the geographical distribution of the primary material must be regarded as satisfactory" (*op. cit.* p. 18), and again: "the agreement (with Census figures) must be regarded as good and the primary material fully representative from the social standpoint" (p. 20).

† Standard Errors are given throughout *The Racial Characters of the Swedish Nation*, and the probable errors in this table were found as those in Table III.

TABL

Mean Values with their Probable Errors for different Secti

(Body measurements are in cms

Divisions		Size of Sample (n)	Stature	Supra-sternal Height	Trunk Length	Arm Length	Leg Length	Bi-achromial Diameter	Inter-iliocristal Breadth	T Le I
Sections	1	2,993	171·34 ± ·07	140·26 ± ·07	52·46 ± ·03	78·50 ± ·04	91·30 ± ·05	39·36 ± ·02	28·44 ± ·02	30·6
	2	1,059	171·30 ± ·12	140·17 ± ·11	52·33 ± ·05	78·40 ± ·07	91·34 ± ·09	39·22 ± ·03	28·45 ± ·03	30·6
	3	406	169·86 ± ·19	138·86 ± ·18	52·13 ± ·08	77·22 ± ·10	90·32 ± ·14	38·89 ± ·05	28·39 ± ·05	30·7
	4	337	171·67 ± ·23	140·38 ± ·19	52·21 ± ·08	77·64 ± ·12	91·67 ± ·17	38·90 ± ·06	28·37 ± ·05	30·5
	5	7,054	172·50 ± ·05	141·12 ± ·04	52·39 ± ·02	79·00 ± ·03	92·23 ± ·03	39·50 ± ·01	29·01 ± ·01	30·4
	6	3,200	172·45 ± ·07	141·13 ± ·06	52·47 ± ·03	78·85 ± ·04	92·16 ± ·05	39·39 ± ·02	28·99 ± ·02	30·5
	7	1,245	172·01 ± ·11	140·64 ± ·10	52·38 ± ·04	78·11 ± ·06	91·76 ± ·08	39·14 ± ·03	28·83 ± ·03	30·5
	8	1,723	172·31 ± ·10	140·87 ± ·08	52·53 ± ·04	77·85 ± ·05	91·84 ± ·07	39·04 ± ·03	28·75 ± ·02	30·5
	9	6,496	172·35 ± ·05	141·02 ± ·04	52·17 ± ·02	79·01 ± ·03	92·35 ± ·04	39·46 ± ·01	28·92 ± ·01	30·3
	10	4,642	172·19 ± ·06	140·85 ± ·05	52·35 ± ·02	78·60 ± ·03	92·00 ± ·04	39·32 ± ·02	28·89 ± ·02	30·4
	11	1,894	172·28 ± ·09	140·89 ± ·08	52·70 ± ·04	78·11 ± ·05	91·70 ± ·07	39·14 ± ·02	28·76 ± ·02	30·6
	12	2,465	172·49 ± ·08	141·04 ± ·07	52·53 ± ·03	78·04 ± ·05	92·01 ± ·06	39·02 ± ·02	28·69 ± ·02	30·5
	13	3,687	171·92 ± ·06	140·67 ± ·06	52·11 ± ·03	78·57 ± ·04	92·06 ± ·05	39·13 ± ·02	28·87 ± ·02	30·4
	14	2,665	171·78 ± ·08	140·59 ± ·07	52·17 ± ·03	78·36 ± ·04	91·92 ± ·06	39·03 ± ·02	28·84 ± ·02	30·4
	15	625	171·61 ± ·16	140·27 ± ·14	52·17 ± ·07	77·86 ± ·09	91·60 ± ·11	38·95 ± ·05	28·64 ± ·04	30·4
	16	1,737	172·04 ± ·09	140·68 ± ·09	52·44 ± ·04	77·55 ± ·05	91·74 ± ·07	38·76 ± ·03	28·58 ± ·02	30·5
(E)	17	4,755	173·03 ± ·06	141·55 ± ·05	52·62 ± ·02	77·69 ± ·03	92·42 ± ·04	38·89 ± ·02	28·55 ± ·01	30·4
(North)	A	4,795	171·23 ± ·06	140·13 ± ·05	52·39 ± ·02	78·31 ± ·03	91·24 ± ·04	39·19 ± ·02	28·43 ± ·01	30·6
(West)	B	13,222	172·43 ± ·03	141·04 ± ·03	52·43 ± ·01	78·73 ± ·02	92·12 ± ·02	39·38 ± ·01	28·95 ± ·01	30·4
(East)	C	15,497	172·32 ± ·03	140·96 ± ·03	52·35 ± ·01	78·62 ± ·02	92·11 ± ·02	39·31 ± ·01	28·86 ± ·01	30·4
(South)	D	8,714	171·88 ± ·04	140·62 ± ·04	52·20 ± ·02	78·25 ± ·02	91·92 ± ·03	39·01 ± ·01	28·79 ± ·01	30·4
(Agric.)	α	20,230	172·17 ± ·03	140·87 ± ·03	52·27 ± ·01	78·85 ± ·02	92·09 ± ·02	39·39 ± ·01	28·87 ± ·01	30·4
(Mixed)	β	11,566	172·10 ± ·04	140·80 ± ·03	52·33 ± ·02	78·59 ± ·02	91·96 ± ·03	39·26 ± ·01	28·86 ± ·01	30·4
(Industr.)	γ	4,170	171·86 ± ·06	140·52 ± ·06	52·46 ± ·02	77·98 ± ·03	91·56 ± ·04	39·09 ± ·02	28·72 ± ·02	30·6
(Urban)	δ	6,262	172·27 ± ·05	140·85 ± ·05	52·48 ± ·02	77·83 ± ·03	91·87 ± ·04	38·95 ± ·01	28·65 ± ·01	30·3

* Standard Errors are given throughout *The Racial Characters of the Swedish Nation*; the Probable Errors in the present table were
Neither were available for the cases of the Supra-sternal Height and the Bi-acromial Index, and for these it was assumed that the Sta

TABLE III.

Probable Errors for different Sections (1—17), *Territories and Occupational Classes of Sweden**.

(Body measurements are in cms. and head measurements in mms.)

Length	Bi-achromial Diameter	Inter-iliocristal Breadth	Trunk Length Index	Leg Length Index	Bi-acromial Index	Head Length	Head Breadth	Face Bread
0 ± ·05	39·36 ± ·02	28·44 ± ·02	30·69 ± ·02	53·27 ± ·02	23·00 ± ·011	193·74 ± ·07	152·64 ± ·06	137·53 ± ·
4 ± ·09	39·22 ± ·03	28·45 ± ·03	30·63 ± ·02	53·31 ± ·03	22·91 ± ·019	192·95 ± ·13	151·93 ± ·11	137·06 ± ·
2 ± ·14	38·89 ± ·05	28·39 ± ·05	30·79 ± ·04	53·10 ± ·04	22·90 ± ·031	191·90 ± ·20	151·47 ± ·17	135·79 ± ·
7 ± ·17	38·90 ± ·06	28·37 ± ·05	30·52 ± ·04	53·38 ± ·05	22·70 ± ·034	192·61 ± ·23	151·03 ± ·18	135·15 ± ·
3 ± ·03	39·50 ± ·01	29·01 ± ·01	30·45 ± ·01	53·47 ± ·01	22·92 ± ·007	194·69 ± ·05	150·48 ± ·04	135·92 ± ·
6 ± ·05	39·39 ± ·02	28·99 ± ·02	30·51 ± ·01	53·44 ± ·02	22·86 ± ·011	194·54 ± ·07	150·13 ± ·06	135·74 ± ·
6 ± ·08	39·14 ± ·03	28·83 ± ·03	30·52 ± ·02	53·34 ± ·02	22·78 ± ·018	194·36 ± ·12	149·98 ± ·10	135·34 ± ·
4 ± ·07	39·04 ± ·03	28·75 ± ·02	30·56 ± ·02	53·32 ± ·02	22·67 ± ·015	193·91 ± ·10	149·55 ± ·08	135·07 ± ·
5 ± ·04	39·46 ± ·01	28·92 ± ·01	30·35 ± ·01	53·58 ± ·01	22·92 ± ·008	194·12 ± ·05	150·29 ± ·04	136·12 ± ·
0 ± ·04	39·32 ± ·02	28·89 ± ·02	30·48 ± ·01	53·43 ± ·01	22·86 ± ·009	193·74 ± ·06	150·26 ± ·05	135·89 ± ·
0 ± ·07	39·14 ± ·02	28·76 ± ·02	30·66 ± ·02	53·33 ± ·02	22·75 ± ·014	193·59 ± ·10	149·93 ± ·08	135·63 ± ·
1 ± ·06	39·02 ± ·02	28·69 ± ·02	30·53 ± ·02	53·31 ± ·02	22·66 ± ·013	193·54 ± ·08	149·84 ± ·07	135·47 ± ·
6 ± ·05	39·13 ± ·02	28·87 ± ·02	30·40 ± ·01	53·50 ± ·01	22·79 ± ·010	193·62 ± ·07	150·95 ± ·06	136·71 ± ·
2 ± ·06	39·03 ± ·02	28·84 ± ·02	30·45 ± ·02	53·44 ± ·02	22·74 ± ·012	193·01 ± ·08	150·71 ± ·07	136·41 ± ·
0 ± ·11	38·95 ± ·05	28·64 ± ·04	30·48 ± ·03	53·36 ± ·03	22·72 ± ·025	193·13 ± ·16	150·14 ± ·14	136·43 ± ·
4 ± ·07	38·76 ± ·03	28·58 ± ·02	30·56 ± ·02	53·29 ± ·02	22·57 ± ·015	193·29 ± ·10	150·31 ± ·08	136·00 ± ·
2 ± ·04	38·89 ± ·02	28·55 ± ·01	30·49 ± ·01	53·43 ± ·01	22·50 ± ·009	193·34 ± ·06	149·73 ± ·05	135·38 ± ·
4 ± ·04	39·19 ± ·02	28·43 ± ·01	30·67 ± ·01	53·27 ± ·01	22·95 ± ·009	193·33 ± ·06	152·27 ± ·05	137·11 ± ·
2 ± ·02	39·38 ± ·01	28·95 ± ·01	30·48 ± ·01	53·43 ± ·01	22·90 ± ·005	194·52 ± ·04	150·23 ± ·03	135·71 ± ·
1 ± ·02	39·31 ± ·01	28·86 ± ·01	30·46 ± ·01	53·45 ± ·01	22·76 ± ·005	193·85 ± ·03	150·14 ± ·03	135·89 ± ·
2 ± ·03	39·01 ± ·01	28·79 ± ·01	30·45 ± ·01	53·48 ± ·01	22·72 ± ·007	193·33 ± ·04	150·69 ± ·04	136·45 ± ·
9 ± ·02	39·39 ± ·01	28·87 ± ·01	30·44 ± ·01	53·48 ± ·01	22·90 ± ·004	194·17 ± ·03	150·82 ± ·02	136·36 ± ·
6 ± ·03	39·26 ± ·01	28·86 ± ·01	30·49 ± ·01	53·42 ± ·01	22·83 ± ·006	193·72 ± ·04	150·48 ± ·03	136·07 ± ·
6 ± ·04	39·09 ± ·02	28·72 ± ·02	30·60 ± ·01	53·31 ± ·01	22·76 ± ·010	193·58 ± ·07	150·12 ± ·05	135·67 ± ·
7 ± ·04	38·95 ± ·01	28·65 ± ·01	30·54 ± ·01	53·31 ± ·01	22·63 ± ·008	193·52 ± ·05	149·95 ± ·04	135·48 ± ·

Probable Errors in the present table were calculated from the Standard Deviations provided in the above book, or from va
and for these it was assumed that the Standard Deviations for each component distribution were equal to the approxima

of Sweden.*

adth	Face Breadth	Morphological Face Height	Minimum Frontal Diameter	Cephalic Index	Morpho-logical Face Index
·06	137·53 ± ·06	127·90 ± ·09	105·26 ± ·05	78·86 ± ·04	93·10 ± ·07
·11	137·06 ± ·10	127·67 ± ·16	105·02 ± ·09	78·82 ± ·07	93·26 ± ·13
·17	135·79 ± ·16	125·86 ± ·24	104·53 ± ·15	79·01 ± ·11	92·81 ± ·20
·18	135·15 ± ·17	125·91 ± ·27	104·47 ± ·16	78·47 ± ·12	93·26 ± ·21
·04	135·92 ± ·04	127·35 ± ·06	104·53 ± ·03	77·38 ± ·02	93·77 ± ·04
·06	135·74 ± ·06	126·94 ± ·08	104·54 ± ·05	77·25 ± ·04	93·59 ± ·07
·10	135·34 ± ·09	126·76 ± ·12	104·70 ± ·08	77·25 ± ·06	93·75 ± ·10
·08	135·07 ± ·08	126·11 ± ·11	104·20 ± ·07	77·20 ± ·05	93·45 ± ·09
·04	136·12 ± ·04	126·74 ± ·06	104·48 ± ·04	77·49 ± ·03	93·18 ± ·05
·05	135·89 ± ·05	126·30 ± ·07	104·29 ± ·04	77·64 ± ·03	93·00 ± ·06
·08	135·63 ± ·07	125·66 ± ·10	104·00 ± ·07	77·41 ± ·05	92·76 ± ·09
·07	135·47 ± ·07	125·56 ± ·09	104·19 ± ·06	77·50 ± ·04	92·78 ± ·07
·06	136·71 ± ·05	126·61 ± ·08	105·13 ± ·05	78·02 ± ·03	92·69 ± ·06
·07	136·41 ± ·06	126·16 ± ·09	104·94 ± ·06	78·18 ± ·04	92·56 ± ·07
·14	136·43 ± ·13	126·32 ± ·18	105·16 ± ·12	77·82 ± ·08	92·70 ± ·15
·08	136·00 ± ·08	126·26 ± ·11	104·81 ± ·07	77·84 ± ·05	92·92 ± ·09
·05	135·38 ± ·05	125·61 ± ·07	104·21 ± ·04	77·53 ± ·03	92·87 ± ·05
·05	137·11 ± ·05	127·53 ± ·07	105·09 ± ·04	78·84 ± ·03	93·12 ± ·06
·03	135·71 ± ·03	127·03 ± ·04	104·50 ± ·03	77·31 ± ·02	93·68 ± ·03
·03	135·89 ± ·03	126·29 ± ·04	104·32 ± ·02	77·53 ± ·02	93·01 ± ·03
·04	136·45 ± ·03	126·38 ± ·05	105·01 ± ·03	78·02 ± ·02	92·69 ± ·04
·02	136·36 ± ·02	127·10 ± ·03	104·92 ± ·02	77·75 ± ·01	93·28 ± ·03
·03	136·07 ± ·03	126·57 ± ·04	104·57 ± ·03	77·76 ± ·02	93·08 ± ·04
·05	135·67 ± ·05	126·10 ± ·07	104·43 ± ·05	77·58 ± ·03	93·05 ± ·06
·04	135·48 ± ·04	125·92 ± ·06	104·37 ± ·04	77·56 ± ·03	93·02 ± ·05

book, or from values calculated from the frequency distributions themselves.
to the approximate values calculated for the total sample (see p. 97).

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TABL

Reduced Coefficients

(*a*) Coefficients for Sections.

	1							
A (North) 2	0·13 ± ·015	2					B (West)	C (Ea
3	2·41 ± ·032	1·63 ± ·039	3				2·41 ± ·003	2·08 ±
4	2·05 ± ·038	0·99 ± ·045	1·24 ± ·063	4		A	B	0·16 ±
5	2·75 ± ·006	2·33 ± ·013	5·98 ± ·030	2·42 ± ·036	5			
B (West) 6	2·91 ± ·008	2·43 ± ·015	4·77 ± ·032	2·12 ± ·038	0·05 ± ·005	6		C
7	2·79 ± ·013	2·03 ± ·020	3·60 ± ·038	1·12 ± ·044	0·58 ± ·011	0·31 ± ·013	7	
8	3·61 ± ·011	2·64 ± ·018	3·69 ± ·035	1·04 ± ·041	1·12 ± ·008	0·67 ± ·010	0·12 ± ·016	8
9	2·78 ± ·006	2·19 ± ·013	4·84 ± ·030	2·39 ± ·036	0·15 ± ·003	0·21 ± ·005	0·70 ± ·011	1·15 ±
C (East) 10	2·30 ± ·006	1·64 ± ·013	3·76 ± ·031	1·17 ± ·037	0·33 ± ·004	0·20 ± ·006	0·30 ± ·012	0·51 ±
11	2·89 ± ·010	2·10 ± ·017	3·25 ± ·035	1·01 ± ·040	1·09 ± ·008	0·67 ± ·010	0·35 ± ·015	0·15 ±
12	3·11 ± ·009	2·19 ± ·016	3·39 ± ·033	0·77 ± ·039	1·17 ± ·006	0·75 ± ·008	0·34 ± ·014	0·10 ±
13	1·50 ± ·007	0·98 ± ·014	3·33 ± ·032	1·12 ± ·037	0·81 ± ·005	0·80 ± ·007	0·83 ± ·012	1·32 ±
D (South) 14	1·61 ± ·008	0·91 ± ·015	2·51 ± ·033	0·63 ± ·039	1·17 ± ·006	1·03 ± ·008	0·79 ± ·014	1·07 ±
15	1·82 ± ·022	1·02 ± ·029	1·98 ± ·047	0·94 ± ·053	1·56 ± ·020	1·26 ± ·022	0·56 ± ·028	0·72 ±
16	2·24 ± ·011	1·39 ± ·018	2·35 ± ·035	0·36 ± ·041	1·91 ± ·008	1·50 ± ·010	0·61 ± ·016	0·51 ±
E (Cities) 17	3·88 ± ·006	2·85 ± ·013	4·24 ± ·031	0·95 ± ·037	1·77 ± ·004	1·30 ± ·006	0·81 ± ·012	0·36 ±
	1	2	3	4	5	6	7	8
	A (North)				B (West)			

TABLE V.

Reduced Coefficients of Racial Likeness.

(*b*) Coefficients for Territories and Occupational Classes.

		B (West)	C (East)	D (South)	E (Cities)	*a* (Agric.)	*β* (Mixed)	*γ* (Industrial)	*δ* (Urban)	
	A	2·41 ± ·003	2·08 ± ·003	1·18 ± ·004	3·13 ± ·005	—	—	—	—	A (North)
5	B	0·16 ± ·002	0·84 ± ·002	1·25 ± ·003	—	—	—	—	—	B (West)
5 ± ·005	6	C	0·43 ± ·002	0·80 ± ·003	—	—	—	—	—	C (East)
8 ± ·011	0·31 ± ·013	7	D	0·99 ± ·004	—	—	—	—	—	D (South)
2 ± ·008	0·67 ± ·010	0·12 ± ·016	8	E	1·56 ± ·003	0·97 ± ·003	0·60 ± ·005	0·23 ± ·004	E (Cities)	
5 ± ·003	0·21 ± ·005	0·70 ± ·011	1·15 ± ·008	9	*a*	0·11 ± ·002	0·78 ± ·003	1·03 ± ·002	*a* (Agricultural)	
3 ± ·004	0·20 ± ·006	0·30 ± ·012	0·51 ± ·009	0·21 ± ·004	10	*β*	0·30 ± ·004	0·49 ± ·003	*β* (Mixed)	
9 ± ·008	0·67 ± ·010	0·35 ± ·015	0·15 ± ·013	1·05 ± ·008	0·33 ± ·009	11	*γ*	0·08 ± ·005	*γ* (Industrial)	
7 ± ·006	0·75 ± ·008	0·34 ± ·014	0·10 ± ·011	1·01 ± ·006	0·36 ± ·007	0·06 ± ·011	12	*δ*		
1 ± ·005	0·80 ± ·007	0·83 ± ·012	1·32 ± ·010	0·50 ± ·005	0·39 ± ·006	1·07 ± ·009	0·96 ± ·008	13		
7 ± ·006	1·03 ± ·008	0·79 ± ·014	1·07 ± ·011	0·79 ± ·006	0·39 ± ·007	0·79 ± ·010	0·67 ± ·009	0·09 ± ·007	14	
3 ± ·020	1·26 ± ·022	0·56 ± ·028	0·72 ± ·025	1·21 ± ·020	0·61 ± ·021	0·66 ± ·024	0·51 ± ·023	0·38 ± ·022	0·14 ± ·023	15
1 ± ·008	1·50 ± ·010	0·61 ± ·016	0·51 ± ·013	1·72 ± ·008	0·87 ± ·009	0·53 ± ·013	0·34 ± ·011	0·86 ± ·010	0·51 ± ·011	0·11 ± ·025	16
7 ± ·004	1·30 ± ·006	0·81 ± ·012	0·36 ± ·009	1·53 ± ·004	0·89 ± ·005	0·45 ± ·009	0·18 ± ·007	1·45 ± ·006	1·12 ± ·007	0·92 ± ·021	0·50 ±
5	6	7	8	9	10	11	12	13	14	15	16
B (West)				C (East)				D (South)			

To face p. 9

North)	
West)	
East)	
South)	
Cities)	
Agricultural)	
Mixed)	
Industrial)	
15	
± ·025	16
? ± ·021	0·50 ± ·009
15	16

To face p. 97

sternal Height, and Leg Length Index were not given by the authors. They were obtained indirectly in the way explained below. The Bi-acromial Index is defined as the ratio of the Bi-acromial Diameter to the Stature, i.e.,

$$\text{Bi-acromial Index } (z) = 100\,\frac{\text{Bi-acromial diameter } (x)}{\text{Stature } (y)}.$$

Therefore writing r_{xy} as the correlation between Bi-acromial Diameter (x) and Stature (y), we have approximately:

$$v_z^2 = v_x^2 + v_y^2 - 2r_{xy}\,v_x v_y,$$

where v_z, v_x, v_y are the coefficients of variation ($100\,\sigma/M$) for the Bi-acromial Index, Bi-acromial Diameter and Stature respectively. Substituting the constants for the total sample (Table IV):

$$M_y = 172{\cdot}23, \quad \sigma_y = 5{\cdot}93, \quad v_y = 3{\cdot}44, \quad r_{xy} = +0{\cdot}47,$$
$$M_x = 39{\cdot}23, \quad \sigma_x = 1{\cdot}67, \quad v_x = 4{\cdot}26, \quad M_z = 22{\cdot}80,$$

we obtain $\sigma_z = 0{\cdot}92$ approximately.

Again the authors define* (p. 73)

$$\text{Supra-sternal Height } (z) = \text{Trunk Length } (x) + \text{Leg Length } (y) - 3{\cdot}5 \text{ cm.}$$

so that $$\sigma_z^2 = \sigma_x^2 + \sigma_y^2 + 2r_{xy}\sigma_x\sigma_y.$$

Since $$\sigma_x = 2{\cdot}41, \quad \sigma_y = 4{\cdot}30, \quad \text{and} \quad r_{xy} = +0{\cdot}18$$

we have $\sigma_z^2 = 28{\cdot}0288$, and $\sigma_z = 5{\cdot}29$ approximately.

Finally, the standard deviation for Leg Length Index was directly calculated from the frequency distribution given in the Swedish work, Table VIII (Supplement, p. 34).

The authors state that the measurements were taken to the nearest millimetre with an "Anthropometer" (compass callipers or Tasterzirkel), and sliding callipers (Gleitzirkel) supplied by P. Hermann of Zürich.

The following notes on measurements are given (pp. 10—11):

"*Bi-acromial Diameter*, defined as the distance between the acromial points, is measured, in departure from the instructions given in Martin†, from the back, the immediate reason for this being to control the posture during measurement."

"*Morphological Face Height* must be regarded as less exactly determined than the other measurements of the head, since the examiner often cannot locate the nasion (sutura naso-frontalis) with certainty."

"*Trunk Length* was calculated as the difference between the supra-sternal height and the height of symphysion."

"*Height of Symphysion* (upper border of symphysis pubis in the middle line). Measurement is rendered difficult in rare cases of excessive corpulence."

"*Arm Length* is the difference between the height of acromion and the height of dactylion, and *Leg Length* is obtained by adding 35 mm. to the height of symphysion (all according to Martin†)."

* [An arbitrary definition, which does not allow for personal or racial variation. ED.]

† It is clear from the remarks under "Bi-acromial Diameter" and "Leg Length" that Martin's directions (presumably those given in his *Lehrbuch der Anthropologie*, 1st edition, 1914) were followed in all cases unless otherwise mentioned.

"*Height of Acromion* could not always be determined with accuracy, since the processus acromialis sometimes showed malformation or at least considerable deviations from the normal form."

"*Height of Dactylion* presented difficulties of measurement in certain cases when the subject could not fully extend the right arm, also when malformations existed in the fingers of the right hand. In such cases this measurement was made from the left."

The *Bi-acromial, Leg Length, Trunk Length* and *Arm Length Indices* all have the stature as the denominator.

2. *Comparisons by the Method of the Coefficient of Racial Likeness.* The main object of the present paper, as I have said, is to present the results of comparisons between the various groups of the Swedish material, described above, made by Professor Karl Pearson's method of the Coefficient of Racial Likeness. In *The Racial Characters of the Swedish Nation* detailed comparisons are made between the means and standard deviations of the characters considered singly, and between the correlations for some pairs of characters calculated for different divisions of the total population. These correlations are shown to be remarkably constant and few significant differences in variability are observed. The means are less constant, and it was felt that a far clearer conception of the anthropological significance of these differences would be given by a generalised criterion, which takes into account a number of characters at the same time, than by the more usual method which deals with individual characters. The coefficient of racial likeness has been extensively used in craniometric work, but little has yet been done in applying it to measurements on the living. One of the principal objections against its use in this case has been the fact that the technique of measurement has not been standardised satisfactorily, and thus the data provided by different observers can seldom be compared with safety*. Such an objection does not apply to the material now under consideration; it constitutes the most complete and most valuable description of the population of a single country which has hitherto been provided. Numbers of individuals large enough to form statistically adequate samples are dealt with, which unfortunately can seldom be the case for cranial series. The number of characters determined is less satisfactory as we can only use 17, and intra-racial correlations between some pairs of these are known to be high. The problem of determining a sufficient number of head and body measurements which are all uncorrelated, or at least lowly correlated, with one another is yet unsolved, and the characters which are customarily determined have certainly not been chosen with this object in view.

If m_s is the mean of the sth character in the first group, σ_s its standard deviation and n the size of the sample, while m_s', σ_s' and n' are the corresponding

* See P. C. Mahalanobis: "On the Need for Standardisation in Measurements on the Living," *Biometrika*, Vol. xx^A. (1928), pp. 1—31.

quantities for the second sample, then Professor Pearson's coefficient of racial likeness is defined to be

$$S\left\{\frac{1}{M}\frac{(m_s-m_s')^2}{\frac{\sigma_s^2}{n}+\frac{\sigma_s'^2}{n'}}\right\}-1+\frac{1}{M}\pm\cdot67449\sqrt{\frac{2}{M}}\quad\ldots\ldots\ldots\ldots\ldots(1),$$

where there are M characters compared*. If pairs of samples are drawn from the same population the coefficients between them will vary round zero with the probable error shown. In the present investigation the number of characters used (17) is the same in every comparison and the term $\frac{1}{M}(=\cdot06)$ has been neglected. The standard deviations are those of the samples and when these are small, as in craniometric work, it has been customary to suppose that they are equal to one another and to the values available for the longest related racial series. The constants have been provided for the Swedish data and they are practically identical for different sections in the case of a particular character and also equal to the general standard deviations calculated for the total sample of 46,983. If the last be denoted by $\bar{\sigma}_s$, then the calculation is greatly simplified by assuming that $\sigma_s=\sigma_s'=\bar{\sigma}_s$. The coefficient becomes

$$S\left(\frac{1}{M}\frac{nn'}{n+n'}\frac{(m_s-m_s')^2}{\bar{\sigma}_s^2}\right)-1\pm\cdot67449\sqrt{\frac{2}{M}}\quad\ldots\ldots\ldots\ldots\ldots(2),$$

when the term $\frac{1}{M}$ is neglected. This is the form which has been used. Values of $\bar{\sigma}$ for the 17 characters are given in Table IV. The characters used should theoretically be uncorrelated with one another, but this condition is far from being satisfied. We are dealing with five indices and the two component lengths from which each is derived are used in addition. The spurious correlations in such cases are probably all greater than 0·5. A number of the absolute measurements also cover one another. Several of the correlations are given in *The Racial Characters of the Swedish Nation*, and in the case of stature and leg length the values for five groups and for the total sample are between 0·86 and 0·88. If the condition were made that no pairs of the measurements used should have correlations greater than 0·5 with one another, then all except three or four of the 17 would have to be rejected. The inclusion of highly correlated measurements is necessitated if the Swedish material is to be dealt with by the method of the coefficient of racial likeness, although these high correlations are far from satisfactory. The procedure is partly justified, perhaps, by the fact that precisely the same group of characters is used in every case. The comparison of these coefficients of racial likeness with others calculated for a different group of measurements would not be justified.

The coefficient provides a measure of the probability that the two samples compared were drawn from the same population. This probability will depend on the sizes of the samples available. It has been suggested that comparable measures of

* Karl Pearson: "On the Coefficient of Racial Likeness," *Biometrika*, Vol. XVIII. (1926), pp. 105—117.

the absolute divergences of the populations represented by the samples may be obtained by reducing each coefficient to the value it would have if each sample were of a standard size*. In the present paper the coefficients have been reduced to values they would have had if each series in the comparison had contained 100 individuals. These values are given by

$$50 \times \frac{\overline{n}+\overline{n}'}{\overline{n}\overline{n}'}\left\{S\left(\frac{1}{M}\frac{nn'}{n+n'}\frac{(m_s-m_s')^2}{\overline{\sigma}_s^{\,2}}\right)-1\right\} \pm \cdot 67449 \times 50 \times \frac{\overline{n}+\overline{n}'}{\overline{n}\overline{n}'}\sqrt{\frac{2}{M}} \ldots (3).$$

Crude coefficients of racial likeness, calculated from formula (2), were first found for the 17 sections defined in Table I and for the territories and occupational classes defined in Table II. The occupational samples for the whole country were made up by pooling the relevant sub-groups of the four major territorial divisions and hence some of the larger groups are not independent samples. No comparisons were made in such cases. Every crude coefficient differs significantly from zero. The values for the sections range from $1{\cdot}04 \pm {\cdot}23$ to $142{\cdot}54 \pm {\cdot}23$ and the mean of the 136 coefficients is 24·85. The values between the territories and occupational classes range from $3{\cdot}99 \pm {\cdot}23$ to $169{\cdot}34 \pm {\cdot}23$ and the mean of the 20 coefficients is 68·22. The difference between these means must be attributed to the fact that the samples are larger in the one case than in the other. All the samples are large and hence it is not surprising to find that the majority of the coefficients are of an order which would indicate marked racial divergence if found for short cranial series. The coefficients clearly increase with the sizes of the series compared and no direct comparison can be made between them until correction is made for this varying factor.

Reduced coefficients of racial likeness calculated from formula (3) are given in Table V*a* for the 17 sections and in Table V*b* for the five territories and four occupational classes. The reduction when all the means are supposed to be based on 100 individuals only is very great in all cases, and values as low as many shown have seldom been found for cranial comparisons. All differ significantly from zero. The 136 coefficients between the sections range from $0{\cdot}05 \pm {\cdot}005$ to $5{\cdot}98 \pm {\cdot}030$ and their mean is 1·38: the 20 coefficients between the territories and occupational classes range from $0{\cdot}08 \pm {\cdot}005$ to $3{\cdot}13 \pm {\cdot}005$ and their mean is 0·97. All the lowest reduced coefficients between the sections are indicated in Fig. I. These measures of relationship suggest an arrangement of the territories (Table V*b*) which is almost linear. The north and west divisions occupy extreme positions, with the east very close to the west and the south closest to the east and rather closer to the west than to the north†. A comparison of the sections of the territories representing any one particular occupational class leads to almost precisely the same geographical arrangement. The urban sections Nos. 4, 16, 12 and 8 are of the north, south, east and west territories respectively, and their reduced coefficients of racial likeness (Table V*a*) give the same arrangement as the total samples for the territories, except

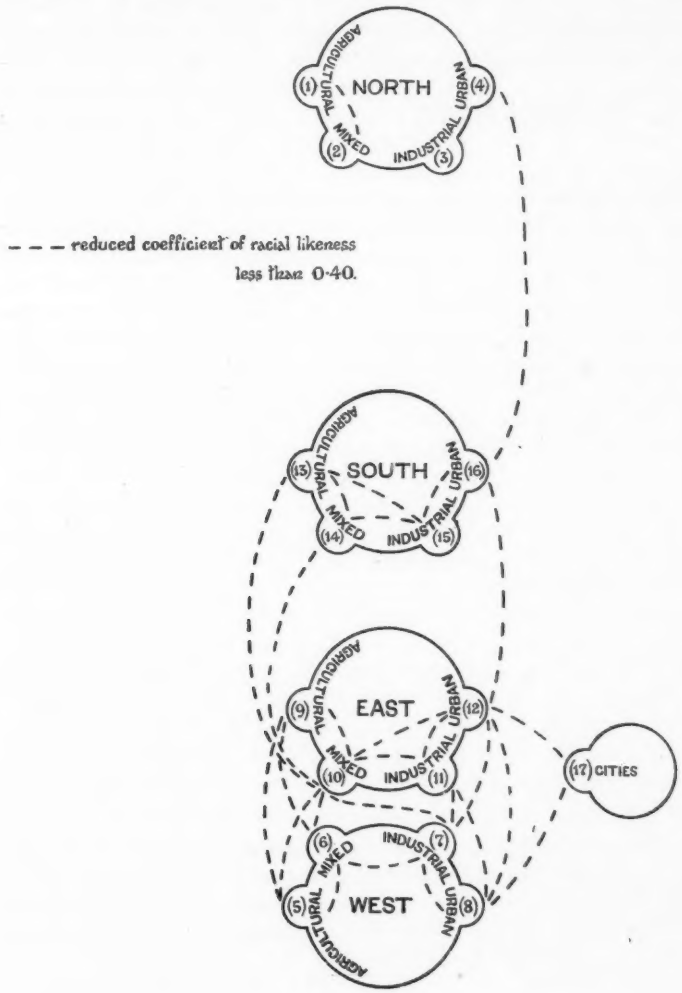
* Karl Pearson: "Note on Standardisation of Method of using the Coefficient of Racial Likeness," *Biometrika*, Vol. xx^{B}. (1928), pp. 376—378.

† To correspond more exactly to this arrangement the distance between the circles representing the north and south territories in Fig. I should be increased considerably.

that the south is now rather nearer to the north than to the west. The same is found for the industrial sections except that the south is rather nearer to the west than to the east. For the mixed sections the south is again rather nearer to the north than to the west, and the single inversion in the case of the agricultural sections is the slightly closer approach of the north to the west than to the east. In spite of these

FIG. I

INTER-RELATIONSHIPS OF VARIOUS GROUPS OF THE POPULATION OF SWEDEN.



small discrepancies, it is true to say that the best linear arrangement of the territories is precisely the same whether we consider their total samples, or the samples for any single one of the four occupational classes. The underlying geographical, or racial, differences can be appreciated nearly as well by considering one particular class only as by considering the total populations irrespective of class. It does not follow from this fact, of course, that the relationships between the territories are precisely the same for one class as for another and, indeed, the contrary can be easily demonstrated. In the comparison of any pair of the four geographical divisions the reduced coefficient between the urban sections is always less than the coefficient between the sections representing any other occupational class. The mixed sections have the next closest degree of relationship in four out of the six possible comparisons, and in five cases the coefficients between the industrial sections are the greatest found. The classes can thus be arranged fairly definitely in the sequence: urban—mixed—agricultural—industrial, with the first on the average showing the minimum and the last the maximum racial differences. All the most intimate connec tions between the sections are shown in Fig. I, the upper limit being fixed arbitrarily as a reduced coefficient of 0·40. The territories were arranged by considering their total samples and these closest links are now only found between the sections of contiguous territories, and there are far more of them between the east and west sections than between those of the east and south, or south and north territories. A comparison of the total occupational samples for the whole of Sweden (Table V*b*, facing p. 97) gives the definite sequence: agricultural—mixed—industrial—urban, and precisely the same order is given by the sections within any one of the territories west, east or south. The connections between any two classes are approximately the same for all these three. The sections of the north territory have different relationships. The agricultural and mixed sections are still as closely connected as for the other territories, but the resemblances of all other pairs of sections are far less close. The same two also stand nearer to the urban than to the industrial sections and the urban stands nearer to the mixed than to the industrial. It is clear that the occupational arrangement which applies uniformly to the south, east and west territories is different in the case of the north owing to the less homogeneous racial constitution of the last territory. Its agricultural and mixed sections are closely linked to one another and they are distinct from all other samples and must therefore be supposed to contain a peculiarly large proportion of a racial element which is foreign to the bulk of the Swedish population. The industrial community of the north territory also stands apart but the urban is not distinguished in this way (see Fig. I). The racial significance of the observed relationships will be considered later.

3. *Comparisons of Individual Characters.* In making comparisons by the method of the coefficient of racial likeness it has been constantly observed that on the average the differences between the various characters vary greatly in significance. The values of the α's* have been used in examining this point, but one

* $\alpha = \frac{nn'}{n+n'} \left(\frac{m_s - m_s'}{\sigma_s} \right)^2$.

objection to their use is that they are influenced, like the coefficients, by the sizes of the samples compared. Since all characters for any one of our samples are based on the same number of individuals it was not necessary to calculate the individual a's in the present investigation. A more direct method of grading the characters can be employed, however. In Table VI are given the inter-group standard deviations (Σ) for the 17 sections of the Swedish material. The co-group standard deviations ($\bar{\sigma}$) in the same table are the general values given for the total sample

TABLE VI.

Inter- and Co-Group Standard Deviations with their Probable Errors.

Character	Inter-group Standard Deviation (Σ)*	Co-group Standard Deviation ($\bar{\sigma}$)†	$\frac{\Sigma}{\bar{\sigma}}$
Cephalic Index	·565 ± ·065	3·14 ± ·007	·180
Head Breadth	·804 ± ·093	5·10 ± ·011	·158
Arm Length	·509 ± ·059	3·34 ± ·007	·153
Bi-acromial Index	·132 ± ·015	0·92 ± ·002	·143
Face Breadth	·655 ± ·076	4·84 ± ·011	·135
Inter-iliocristal Breadth ...	·205 ± ·024	1·52 ± ·003	·135
Bi-acromial Diameter ...	·214 ± ·025	1·67 ± ·004	·128
Stature	·680 ± ·079	5·93 ± ·013	·115
Leg Length	·478 ± ·055	4·30 ± ·009	·111
Head Length	·686 ± ·079	6·19 ± ·014	·111
Supra-sternal Height ...	·569 ± ·066	5·29 ± ·012	·108
Morphological Face Height	·688 ± ·080	6·92 ± ·015	·099
Trunk Length Index ...	·106 ± ·012	1·18 ± ·003	·091
Minimum Frontal Diameter	·372 ± ·043	4·33 ± ·010	·086
Leg Length Index	·106 ± ·012	1·29 ± ·003	·082
Trunk Length	·175 ± ·020	2·41 ± ·005	·072
Morphological Face Index...	·368 ± ·043	5·61 ± ·012	·066

of 46,983 individuals and these are almost precisely the same as the values found for any one of the 17 sections. The $\bar{\sigma}$'s are the ones which were used in computing the coefficients of racial likeness. It is clear that the ratio of Σ to $\bar{\sigma}$ will give a measure of the average significance of the differences found for the various characters. The inter-group variability is small compared with the intra-group variability in every case, but there are still marked differences between the measurements in this respect. The cephalic index tends to show more significant differences than any other character and this has been confirmed in the case of several other comparisons of measurements made either on the living or on the skull. It has been usual to find, too, that the head breadth varies more significantly than the head length and much more significantly than the minimum frontal diameter. The stature is less capable of differentiating the groups than several of the other characters. An index, such as the cephalic or bi-acromial, may vary more significantly than either of its component lengths, or the reverse may hold, as for the leg length and morphological face indices.

* These standard deviations are for the means of the 17 sections given in Table III.

† These standard deviations are for the total sample of 46,983 individuals.

The 17 characters may now be considered individually with regard to both geographical position and occupational class. They can be divided into a number of groups by considering whether the order in which each arranges the 17 sections is controlled more by one of these factors than by the other. The cephalic index is extreme in this respect. The four lowest means are for the sections of the west territory, the sections of the east and the urban section (No. 17) follow next and then the four of the south, while the cephalic indices for the sections of the north territory are greater than any others. There is thus a clear distinction between the territories, and they are arranged in the order shown in Fig. I. The maximum difference between the sections of the same territory with extreme cephalic indices is only 4·0 times its probable error (south and east territories) and no significance whatever can be attached to the orders in which the occupational classes are arranged within the territories. This character is clearly controlled by geographical position and there is no evidence of any significant association with occupational class. The bi-acromial index affords an example of a measurement which is affected by conditions entirely different from these. The order in which the means arrange the 17 sections appears to have no geographical significance whatever, but the three highest indices are for agricultural sections, the lowest is for the sample from the four largest cities (No. 17) and the next four lowest are for the other urban sections. For each territory the highest index is for the agricultural section, the second highest for the mixed, the next for the industrial and the lowest for the urban section. The differences between the agricultural and urban sections of the same territory are very significant in every case, being 8·4, 12·2, 16·9 and 15·1 times their probable errors for the north, south, east and west divisions respectively. The bi-acromial index is thus clearly controlled by the occupational class, and there is no evidence of any significant association with geographical position. These two characters are at opposite extremes in so far as they are controlled by one or other of the factors on the basis of which the groupings were made, but in most other cases there is a definite tendency for a measurement to approach one extreme in this respect rather than the other. The orders in which the sections are arranged may be supposed to have been influenced by both factors in the majority of cases. Whenever there is a clear territorial sequence, or the suggestion of such a sequence, it is always: north, south, east and west. Whenever there is a clear occupational sequence common to all the territories, or the suggestion of such a sequence, it is always: agricultural, mixed, industrial and urban. Paying due regard to the significance of the differences, the following classification of the characters can be made:

(*a*) Characters showing markedly significant territorial differences, but no occupational sequence within the territories—cephalic index, stature, supra-sternal height.

(*b*) Characters showing significant territorial differences and a significant occupational sequence within the territories—head breadth, head length, inter-iliocristal breadth.

(*c*) Characters showing a suggestion of territorial differences, but no occupational sequence within the territories—minimum frontal diameter, leg length.

(*d*) Characters showing a faint suggestion of territorial differences and a markedly significant occupational sequence within the territories—face breadth, arm length, bi-acromial diameter.

(*e*) Characters showing a faint suggestion of territorial differences and a significant occupational sequence within the territories—leg length index, trunk length index.

(*f*) Characters showing no territorial differences and a markedly significant occupational sequence within the territories—morphological face height, bi-acromial index.

(*g*) Characters showing no territorial differences and no occupational sequence within the territories—morphological face index, trunk length.

The comparison of individual characters has confirmed in a very satisfactory way the scheme of relationships suggested by the coefficients of racial likeness. It can now be seen that there are marked differences between the characters not only in their average effect on the coefficients, but also according as they are more or less capable of discriminating between regional or occupational samples. The two in group (*g*) above are the only ones which appear to be quite incapable of serving either purpose and these are the two with the lowest values of $\Sigma/\bar{\sigma}$ (see Table VI). By making a suitable selection from the other 15 it would clearly be possible to obtain coefficients which would emphasise the geographical differences and obscure the occupational, while the reverse effect could be obtained by making a different selection. The characters which show territorial differences will be considered again in the next section. There are seven absolute and three indicial measurements which furnish either a significant, or a markedly significant occupational sequence, and for all except one the agricultural section tends to have the greatest mean and the urban the least. The trunk length index is greater for the urban than for the rural populations, but the reverse is found for the bi-acromial diameter and index, the arm length, the head and face breadths, the head length, the morphological face height, the inter-iliocristal breadth and the leg length index. The fact that the first three of the last nine measurements are greater for rural than for urban samples was to be expected. The relations observed in the case of the others suggest that the agricultural workers have skeletons which are broader in all ways and with relatively longer limbs than town dwellers, though no differences between the statures of the groups can be detected. The differences between the extreme means are all very small, however. The bi-acromial diameter provides a more definite occupational sequence than any other absolute measurement, but the largest mean for a section only exceeds the smallest by 7·4 mm. Whether any of the differences between occupational classes are due to use, or whether they are occasioned by selection, cannot be decided from these Swedish data. With smaller samples, or in the case of a more racially heterogeneous population, it would probably be impossible to prove their existence.

4. *The Racial Constitution of the Swedish Population.* The present study is restricted, on the regional side, to a comparison of the four territorial divisions into

which the whole of Sweden was divided, and some important facts may be overlooked by taking such large areas. All the individuals examined were born in Sweden. The remarkable constancy of the coefficients of variation and correlation, provided in *The Racial Characters of the Swedish Nation*, suggests that the populations are now thoroughly hybridised if they once had diverse racial origins. The coefficients of racial likeness suggested the simple linear arrangement shown in Fig. I and the reasonableness of this order was emphasised by finding that all the characters which are capable of making definite distinctions between the territories show the same sequence from the north at one extreme to the west at the other. Of the 17 measurements there are only six which give significant or markedly significant regional differences when a single occupational class is considered. The total means for the territories are given for these in the table below and this comparison is now not quite so convincing since the relative proportions of the different classes are not the same for all the territories.

Territories	Cephalic index	Stature	Supra-sternal height	Head breadth	Head length	Inter-iliocristal breadth
North	78·84	171·23	140·13	152·27	193·33	28·43
South	78·02	171·88	140·62	150·69	193·33	28·79
East	77·53	172·32	140·96	150·14	193·85	28·86
West	77·31	172·43	141·04	150·23	194·52	28·95

Some pairs of these six measurements are lowly correlated with one another and the fact that they provide the same sequence is all the more significant on that account. The coefficients between the sections are so low that it can only be assumed that all divisions of the total population of Sweden belong predominantly to the same racial type. The observed relationships can be explained on the hypothesis that this basic type has been modified slightly, but in different degrees in different territories, by admixture with another race. The north territory was more modified than any other by this means, the south considerably less, the east still less and the west territory may have been unaffected, or modified to a less extent than any other. The racial crossing seems to have resulted in a perfect blending of all the characters for which data are available and those which show no territorial differences may be assumed to have been the same for the two racial types. These are the conclusions suggested by a purely statistical analysis of the material and we may attempt to reconcile them with what is known of the ethnic history of the country. The following particulars are taken from the section written by Rolf Nordenstreng in *The Racial Characters of the Swedish Nation**.

"The Swedes have inhabited their country since later neolithic times. The main body of the prehistoric population seems to have been of rather distinctly Nordic

* "Origin, Growth and Racial Components of the Swedish Nation," *op. cit.*, pp. 41—49 and summary on p. 174.

race, though other types also occur ...The finds from the Bronze Age and the Iron Age do not present any new types, but agree with those from the Stone Age....The early Swedish kingdom did not consist of more than the present central territories about Lake Mälaren; but gradually other parts of the present kingdom were conquered, the people of the Gauts south of the Swedish settlements between the Baltic Sea and the North Sea being the most important of those incorporated into the nation. All these peoples on the Scandinavian Peninsula were Teutons like the Swedes, of much the same race, and using similar languages. Only in the northernmost part of the country lived Lapps, roving since prehistoric times. The Swedish dominion was early extended to territories east of the Baltic, whence in the course of time came an influx of the East Baltic race, especially in a Finnish immigration in the last years of the 16th, and the first half of the 17th century....That the Nordic race has been the chief stock of Sweden's ancient population, as of the present, is beyond all doubt. But as to what extent it was mixed and with which races, we can venture nothing more than a guess....It is not impossible that the East Baltic race is very ancient in this country, more ancient even than the Nordic, but this cannot be proved and is hardly very likely; the possibility should not, however, be wholly dismissed. The most noteworthy support is given by the type demonstrated by Arbo in South Norway and often called 'the blond brachycephal,' a type which reminds one not a little of the Finnish....According to Lönborg's calculations Sweden (except Norrbotten) and parts of East Norway had at the close of the 17th century a Finnish population of between twelve and thirteen thousand persons. This figure is very likely too low, but nevertheless is highly appreciable, considering that Sweden's entire population then amounted to hardly 1½ millions, and that the parts of the country in which the Finns were living were certainly very sparsely populated. As these immigrants were unusually prolific, their offspring undoubtedly increased at a proportionately higher rate than those of the real Swedes....There is also a Lappic race-admixture in the Finnish population of North Sweden....The number of Finnish-speaking persons in Northern Sweden probably amounts at present to about 30,000....How strong the race-mixture with East Baltic blood has been in Sweden is at present impossible to state. But it would hardly be an exaggeration to assert that at least some hundred thousand present-day Swedes and perhaps many more evince more or less East Baltic characters."

The only foreign races which are known to have influenced the population of Sweden to any marked extent since neolithic times are thus the Finns and, to a lesser extent, the inhabitants of the East Baltic states; the Lapps, as far as is known, have lived in the north as long as the country was inhabited at all. All these alien races are closely allied to one another, and, where they differ from the Swedish type, they apparently do so in the same direction as, for example, in possessing higher cephalic indices and shorter statures. The miscegenation with the so-called nordic population must have been extremely thorough, since the variabilities for all sections are almost identically the same. Even in the north, the bulk of the population must be of "nordic" origin, and it is not surprising to find that the effects of slight differences between the types with which admixture was made in

different regions cannot be detected at all by considering large groups of the existing population. Comparisons made in that way only suggest that there was a crossing with a single racial type resulting in a perfect blending of all the characters considered. The alien element is far more evident in the north than in any other territory; it produced a greater effect in the south than in the central regions of Sweden, and the east was slightly more affected than the west.

Note added in proof.

This paper was originally written as an integral part of an empirical study of certain alternative formulae for the measurement of racial divergence. Very extensive and substantial editing of the text was therefore necessary in publishing it in the present form. I am deeply indebted to Dr G. M. Morant for having carried out the editing work much more satisfactorily than I could have done myself. I also wish to acknowledge the help I received from my assistant Mr Sudhir Kumar Banerjee in reducing the statistical material for the paper. P. C. Mahalanobis, Calcutta, 22nd July, 1930.

SKEW BIVARIATE FREQUENCY SURFACES, EXAMINED IN THE LIGHT OF NUMERICAL ILLUSTRATIONS.

By S. J. PRETORIUS, M.Sc.

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I.

A. *Introductory.*

Since the development of the theory of normal correlation by Galton and Dickson (1886) several attempts have been made to describe analytically a distribution of two correlated variables when both variables follow a law of skew variation. These

attempts may be considered as of two types: those that are founded on one or another hypothesis, and those that are purely empirical. This discrimination, however, is not of great practical importance. The superiority of one frequency function over another depends rather on the success with which that function can be applied to graduate data than on the manner in which it originated. The univariate functions that are in common use can, as a rule, be fitted fairly easily. But in the case of bivariate functions, the process of fitting is extremely laborious. Comparative results are rare, and such illustrations as do exist are of little final value. Either the test has not been stringent enough, or the paucity of observations has made it impossible for a dispassionate judgment to be passed on the goodness of fit. In short, the descriptive power of the various surfaces has as yet not been extensively investigated.

The purposes of this study are: (i) to present an account of the surfaces that have been evolved; (ii) to analyse geometrically a few observed distributions, each containing a large number of observations; hence (iii) to put to a practical test some of the hypotheses from which these surfaces have been developed; and (iv) finally, to compare the adequacy of the surfaces by fitting their marginal and partial moment curves to the observations.

B. *Notation and Terminology.*

To avoid unnecessary repetition, a description is given below of the notation and terminology that will be adopted.

The two variables will be denoted by x and y; the total number of observations by N; the number of observations in an x-array of y's and in a y-array of x's by n_x and n_y respectively; any cell frequency by n_{xy}; the usual correlation coefficient between x and y by r; the correlation ratio of y on x and of x on y by η_{yx} and η_{xy} respectively. The ss'th product-moment coefficient calculated from the observations about any origin will be denoted by $\mu'_{ss'}$ and $\nu'_{ss'}$, according as corrections for grouping have or have not been applied; the dashes will be dropped when the origin is the arithmetic mean $(\bar{x}, \bar{y})$. Thus

$$\nu'_{ss'} = \frac{1}{N} \Sigma\Sigma (n_{xy} \,.\, x^s y^{s'}),$$

and

$$\mu_{ss'} = \frac{\iint F(x,y)\,(x-\bar{x})^s\,(y-\bar{y})^{s'}\,dx\,dy}{\iint F(x,y)\,dx\,dy},$$

where $F(x,y)\,dx\,dy$ expresses the probability that x lies between x and $x+dx$, y between y and $y+dy$, and the integrations are taken over the entire surface.

Still following the usual notation, I shall write:

$$q_{ss'} = \frac{\mu_{ss'}}{\sigma_1^{\,s} \,.\, \sigma_2^{\,s'}},$$

$$\beta_{10} = \frac{\mu_{30}^{\,2}}{\mu_{20}^{\,3}}, \quad \beta_{20} = \frac{\mu_{40}}{\mu_{20}^{\,2}},$$

$$\beta_{01} = \frac{\mu_{03}^{\,2}}{\mu_{02}^{\,3}}, \quad \beta_{02} = \frac{\mu_{04}}{\mu_{02}^{\,2}},$$

where $\sigma_1 \equiv \sqrt{\mu_{20}}$ and $\sigma_2 \equiv \sqrt{\mu_{02}}$ are the standard deviations of the x- and y-marginal totals respectively.

The moments of an array distribution will be termed *array moments*; those of a section of a theoretical surface parallel to the zx or zy plane, will be termed *partial moments*. The corrected sth moment coefficient of an x-array of y's about the mean of the array will be denoted by $\mu_s(y)$; that of a y-array of x's by $\mu_s(x)$; the x- and y-array means by $\mu_1'(y)$ and $\mu_1'(x)$ respectively. The same notation will be used for the partial moments, but it must be remembered that in this case the variable assumes "singular" and not "plural" values; this distinction is brought out by the terms introduced above. The curves in which $\mu_1'(y)$, $\sigma_y \equiv \sqrt{\mu_2(y)}$, $\sqrt{\beta_1(y)} \equiv \dfrac{\mu_3(y)}{\mu_2^{\frac{3}{2}}(y)}$ and $\beta_2(y)-3 \equiv \dfrac{\mu_4(y)}{\mu_2^2(y)}-3$ are plotted to x are the *regression, scedastic, clitic and kurtic curves of y on x**. A system is either *homoscedastic* or *heteroscedastic* according as the arrays "are equally scattered about their means," or not.

C. *Historical.*

1. *Writers before Galton.* The normal probability surface discussed by Lagrange, Laplace, Plana, Gauss, Bravais, has little bearing, if any at all, on the theory of correlation. With admirable clarity Pearson† pointed out a few years ago that the quantities Gauss and Bravais were observing, were absolutely independent of one another. Only by the introduction of quantities linearly related to those observed, did the product terms in their expressions arise. In a more recent paper, dealing with Plana's work, Pearson‡ again indicated that the writers on the theory of observations up to the time of Galton were concerned merely with finding a mathematical expression for the probability of the simultaneous occurrence of two or more errors and not with finding a measure of relationship between two variables organically associated.

Galton, on the other hand, started with the conception that the observed quantities are dependent. In studying the inheritance of traits, he developed in

* Pearson originally defined the scedastic curve as the curve in which the ratio of the standard deviation of the array to the standard deviation of the character in the population at large is plotted to position, and the clitic curve as the curve in which the skewness of the array is plotted to position.

For the Pearson Type III curve: $\text{Skewness} = \dfrac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}} = \frac{1}{2}\sqrt{\beta_1}$.

Wicksell calls the curve $y = -\frac{1}{2}\sqrt{\beta_1(y)}$ ($\equiv$ "skewness") the clitic curve, and the curve $y = \frac{1}{8}(\beta_2(y)-3)$ ($\equiv$ "excess") the synagic curve. I prefer, however, to retain Pearson's original term "kurtosis" as expressing that deviation of frequency curves from the normal type which corresponds to forms more or less flat-topped. Further, being concerned merely with how the skewness varies from array to array and not with the degree of skewness of any particular array distribution, I shall omit all constant multiplying factors for $\sqrt{\beta_1(y)}$ and $\beta_2(y)-3$.

† "Notes on the History of Correlation," *Biometrika*, Vol. XIII. 1920—21, pp. 25—45.

‡ "The Contribution of Giovanni Plana to the Normal Bivariate Frequency Surface," *Biometrika*, Vol. XX^A. 1928, pp. 295—298. See also Walker, Helen M.: "The Relation of Plana and Bravais to the Theory of Correlation," *Isis*, Vol. x. No. 34, 1928, pp. 466—484.

a series of papers, from 1877, the ideas of regression and correlation. Dickson*, in 1886, investigated mathematically the system of concentric ellipses that would correspond to the ellipses deduced by Galton in his study of "Regression towards Mediocrity in Hereditary Stature" (1885).

2. *Skew Univariate Distributions.* At the time when Galton developed his theory of correlation, writers on mathematical statistics realised that the univariate normal law of De Moivre and Laplace could not be regarded as a universal law of frequency distribution; the presence of skewness in homogeneous material was certainly as common as that of normality. Attempts to describe analytically this skew variation, led up to the work of Gram, Thiele, Pearson, Edgeworth, Bruns, Charlier, and Kapteyn, to mention only the most prominent contributors.

It would be superfluous to give here more than a short summary of these curves such as will be required for further reference. We may conveniently treat them under the following three divisions:

(*a*) Pearson's system of skew frequency curves derived from the generalised probability equation:

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{x+a}{b_0+b_1x+b_2x^2};$$

(*b*) (i) Edgeworth's generalised law of error,

(ii) The Gram-Charlier Type A and Type B series;

(*c*) The translated, or transformed, curves of Edgeworth, and of Galton and MacAlister, as treated by Pearson and Wicksell.

Only (*b*) and (*c*) will be shortly discussed.

(*b*) (i) Edgeworth† deduces his generalised law of error from a consideration of a large number, n, of elemental frequency groups which satisfy certain conditions. The most important of these conditions are: that selections from different groups are independent of one another; that the chance of obtaining a particular magnitude from one group is independent of previous selections; that $\frac{\mu_p}{\sigma^p}$ is finite for all values of p in the elemental groups. On these assumptions the frequency locus of the aggregate formed, is found to be:

$$F(x)=e^{-\frac{1}{3!}k_3\frac{d^3}{dx^3}+\frac{1}{4!}k_4\frac{d^4}{dx^4}-\cdots}.\phi(x),$$

where

$$\phi(x)=\frac{1}{\sqrt{2\pi}}.e^{-\frac{1}{2}x^2},$$

$$k_3=\sqrt{\beta_1},\quad k_4=\beta_2-3,\ \ldots\ldots$$

* Galton, Francis: "Family Likeness in Stature." With an appendix by J. D. Hamilton Dickson, *Proc. Roy. Soc.* Vol. XL. 1886, pp. 42—73.

† "The Asymmetrical Probability Curve," *Phil. Mag.* Vol. XLI. 1896, pp. 90—99; "The Law of Error," *Camb. Phil. Trans.* Vol. XX. 1905, pp. 36—65, 113—141.

It is further shown that k_{p+2} is of the order $n^{-\frac{1}{2}p}$, so that to an approximation of the order $\frac{1}{n}$:

$$F(x)=\left[1-\frac{k_3}{3!}\frac{d^3}{dx^3}+\frac{k_4}{4!}\frac{d^4}{dx^4}+\frac{1}{2}\left(\frac{k_3}{3!}\right)^2\cdot\frac{d^6}{dx^6}\right]\phi(x) \quad \ldots\ldots\ldots\ldots(1).$$

Introducing Pearson's definition of the tetrachoric functions, viz.,

$$\tau_s(x)=\frac{1}{\sqrt{s!}}\left(-\frac{d}{dx}\right)^{s-1}\cdot\frac{1}{\sqrt{2\pi}}\cdot e^{-\frac{1}{2}x^2},$$

we have from (1):

$$F(x)=\tau_1+\sqrt{\tfrac{2}{3}}\,.\sqrt{\beta_1}\,.\tau_4+\sqrt{\tfrac{5}{24}}\,.(\beta_2-3)\,.\tau_5+\sqrt{\tfrac{35}{36}}\,.\beta_1\,.\tau_7 \quad \ldots\ldots(1)^{bis}.$$

(*b*) (ii) Several other writers—Gram, Bruns, Charlier—have discussed a series almost identical with (1). Starting from the hypothesis of elementary errors, Charlier* deduces two forms of the frequency function, called by him Type A and Type B. Type A is an extension of the De Moivre-Laplace approximation to the binomial; Type B is an extension of the Poisson limit to the binomial. The transition from Type A to Type B cannot be expressed mathematically. Usually Type B is employed when there is a marked asymmetry, while for slightly asymmetrical curves the type can be determined only by trial.

Type A. $$F(x)=\phi(x)+\sum_{p\geq 3}(-1)^p\frac{A_p}{p!}\frac{d^p\phi(x)}{dx^p},$$

where $$A_3=\sqrt{\beta_1}=\lambda_3, \quad A_4=(\beta_2-3)=\lambda_4,$$

$$A_5=\beta_3'-10\sqrt{\beta_1}=\lambda_5, \quad A_6=\beta_4-15\beta_2+30=\lambda_6+10\lambda_3^2,$$

the λ's being the third, fourth, ... semi-invariants and $\beta_3'=\mu_5/\sigma^5$. Expressed in terms of tetrachoric functions:

$$F(x)=\tau_1+\sqrt{\tfrac{2}{3}}\,.\sqrt{\beta_1}\,.\tau_4+\sqrt{\tfrac{5}{24}}\,.(\beta_2-3)\,.\tau_5 \quad \ldots\ldots\ldots\ldots\ldots(2),$$

up to moments of the fourth order only.

The two approximations $(1)^{bis}$ and (2) are not quite identical. It is partly on this ground that Edgeworth† criticised the Gram-Charlier series as not being the true generalisation of Laplace's law of error. In a later paper Charlier‡ has shown the order of magnitude of the coefficients $A_p'\equiv\frac{A_p}{p!}$ to be:

* "Über das Fehlergesetz," *Arkiv för Mat., Astr. och Fysik*, Bd. 2, No. 8, 1905, pp. 1—9; "Über die Darstellung willkürlicher Funktionen," *Arkiv för Mat., Astr. och Fysik*, Bd. 2, No. 20, 1905, pp. 1—22; "Die strenge Form des Bernoulli'schen Theorems," *Arkiv för Mat., Astr. och Fysik*, Bd. 5, No. 15, 1909, pp. 1—22; "Contributions to the Mathematical Theory of Statistics. 5. Frequency Curves of Type A in Heterograde Statistics," *Arkiv för Mat., Astr. och Fysik*, Bd. 9, No. 25, 1914, pp. 1—17.

† "On the Representation of Statistical Frequency by a Series," *Journ. Roy. Stat. Soc.* Vol. LXX. 1907, pp. 102—106.

‡ "Die strenge Form des Bernoulli'schen Theorems," *Arkiv för Mat., Astr. och Fysik*, Bd. 5, No. 15, 1909, pp. 1—22.

$$\left.\begin{array}{lll} A_3' \text{ is of the order of magnitude of} & \dfrac{1}{\sqrt{n}} \\ A_4' \quad \text{,,} \quad \text{,,} \quad \text{,,} & \dfrac{1}{n} \\ A_6' \quad \text{,,} \quad \text{,,} \quad \text{,,} & \dfrac{1}{n} \\ A_5' \quad \text{,,} \quad \text{,,} \quad \text{,,} & \dfrac{1}{n\sqrt{n}} \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3),$$

where n is the number of "error-sources." It is further shown* that λ_6 is of the order $\dfrac{1}{n^2}$, so that

$$\frac{A_6}{6!} = \frac{1}{2}\left(\frac{A_3}{3!}\right)^2 + \text{term of order } \frac{1}{n^2},$$

and hence to an approximation of the order $\dfrac{1}{n}$:

$$F(x) = \tau_1 + \sqrt{\tfrac{2}{3}} \,.\, \sqrt{\beta_1} \,.\, \tau_4 + \sqrt{\tfrac{5}{24}} \,.\, (\beta_2 - 3) \,.\, \tau_5 + \sqrt{\tfrac{35}{36}} \,.\, \beta_1 \,.\, \tau_7,$$

which is identical with $(1)^{bis}$. In the paper just cited, Wicksell shows the orders of magnitude (3) not to be a necessary consequence of the hypothesis of elementary errors; they can be deduced only when the skewness of the error distributions is regarded as independent of n. The convergency of tetrachoric expansions has been discussed from a more practical point of view by Pearson†. He assigns definite values to β_1 and β_2 and demonstrates that, unless the skewness be chiefly in the one direction or the other, any tetrachoric term in the series is not negligible as compared with those preceding it. A good illustration of the oscillatory nature of expansions in terms of tetrachoric functions and of their practical non-convergency, is provided in a paper by James Henderson‡. Closely associated with the problem of convergency, is the appearance of negative frequencies in the tails of the curve and the impossibility of making the curve start at a fixed point. Although a fairly good description of the central part of the observations is likely to be obtained, the curve fails us almost entirely in the determination of the significance of outlying observations.

For convenience we shall refer to equation (2) as Type Aa; to $(1)^{bis}$ as Type Ab; to both or either of the two as Type A.

Type B. $$F(x) = k_0 \,.\, \psi_\lambda(x) - \frac{k_1}{1!}\Delta\psi_\lambda(x) + \frac{k_2}{2!}\Delta^2\psi_\lambda(x) - \ldots,$$

where $$\Delta\psi_\lambda(x) = \psi_\lambda(x) - \psi_\lambda(x-1),$$

$$\Delta^2\psi_\lambda(x) = \psi_\lambda(x) - 2\psi_\lambda(x-1) + \psi_\lambda(x-2),$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

* Wicksell, S. D.: "The Correlation Function of Type A and the Regression of its Characteristics," *Kungl. Sv. Vet. Akad. Handl.* Bd. 58, No. 3, 1917, pp. 1—48.

† "The Fifteen Constant Bivariate Frequency Surface," *Biometrika*, Vol. XVII. 1925, pp. 277—280.

‡ "On Expansions in Tetrachoric Functions," *Biometrika*, Vol. XIV. 1922—23, pp. 157—185.

and $$\psi_\lambda(x) = \frac{e^{-\lambda}}{\pi}\int_0^{+\pi} e^{\lambda\cos w}\,.\cos(\lambda\sin w - xw)\,dw$$

$$= \frac{e^{-\lambda}\,.\,\lambda^m}{m\,!}\,,\quad \text{when } x \text{ is a positive integer, } m.$$

The function $\psi_\lambda(x)$ and the more general function

$$\vartheta_{\lambda,\eta}(x) = \frac{e^{-\lambda}}{\pi}\int_0^{+\pi} e^{\lambda\cos w}\,.\cos(\eta\sin w - xw)\,dw$$

have been introduced by Charlier* as continuous functions representing the Poisson exponential. Charlier confined his treatment to the function $\psi_\lambda(x)$ and determined the coefficients k by an approximate method. The more general function $\vartheta_{\lambda,\eta}(x)$ has been discussed by Jørgensen†. He finds the exact values of the coefficients and considers special cases of a linear transformation of the argument. The order of magnitude of the coefficients necessitates the use of an even number of terms in successive approximations to the series.

Because of the theoretical and practical objections that can be adduced against the use of these continuous generating functions, I shall not give a detailed account of the Type B distribution. In a critical note on Jørgensen's proof of these functions Steffensen‡ has shown that the moment integrals $\int_0^\infty \vartheta(x)\,.\,x^p\,dx$ are divergent. Apart also from the negative frequencies that arise in applying the series, the curve assumes a sinusoidal form for fractional values of the variate: "for brudne Værdier af Abscisserne svinger de i Virkeligheden sinusoideformet og giver for store Abscisser negative Ordinater, hvad der navnlig for $\lambda = \eta$ for negative Abscisser træder stærkt frem§."

(c) *The Method of Translation.* Let $\eta = \phi(\xi)$ be the frequency curve of a hypothetical variate ξ. Replace ξ by a function $f(x)$ of x, x being the quantity observed. If the areas between corresponding ordinates of the generating curve $\phi(\xi)$ and the generated ("translated") curve $F(x)$ are to remain unchanged, then

$$y = F(x) = \phi[f(x)]\,.\,f'(x).$$

By a suitable choice of ϕ and f, the form of $F(x)$ might be such as is commonly observed in practical statistics.

* "Die Zweite Form des Fehlergesetzes," *Arkiv för Mat., Astr. och Fysik*, Bd. 2, No. 15, 1905, pp. 1—8; "Weiteres über das Fehlergesetz," *Arkiv för Mat., Astr. och Fysik*, Bd. 4, No. 13, 1907, pp. 1—9.

† "Note sur la fonction de répartition de Type B de M. Charlier," *Arkiv för Mat., Astr. och Fysik*, Bd. 10, No. 15, 1914, pp. 1—18; *Undersøgelser over Frequensflader og Korrelation.* København: Arnold Busck, 1916.

‡ *Svenska Aktuarieföreningens Tidskrift*, Nos. 4—5, 1916. See also *Matematisk Iagttagelselære*, København, 1923, p. 71.

§ Jørgensen: *loc. cit.* p. 28.

(i) *Edgeworth's Translated Curves**. Take $\phi(\xi)$ to be the normal curve:

$$\eta = \frac{1}{\sqrt{\pi}} . e^{-\xi^2},$$

and consider the particular equation of translation:

$$x = a(\xi + k\xi^2 + \lambda\xi^3) \quad \text{.....(4).}$$

The ordinate of the translated curve is:

$$y = \frac{1}{\sqrt{\pi}} . e^{-\xi^2} . \frac{1}{a(1 + 2k\xi + 3\lambda\xi^2)} \quad \text{.....(5),}$$

and the rth moment about the origin—the median—is:

$$M_r' = \int_{-\infty}^{+\infty} y . x^r dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} a^r (\xi + k\xi^2 + \lambda\xi^3)^r . e^{-\xi^2} d\xi.$$

From the moment coefficients about the mean, the constants a, k, and λ can be determined. The equations are:

$$\left.\begin{aligned} &\chi(\tfrac{3}{2} + \chi + 9\lambda + \tfrac{135}{8}\lambda^2)^2 - j^2(1 + \chi + 3\lambda + \tfrac{15}{4}\lambda^2)^3 = 0 \\ &6\chi + 3\lambda + 3\chi^2 + 54\chi\lambda + 27\lambda^2 + 135\chi\lambda^2 + \tfrac{405}{4}\lambda^3 + \tfrac{1215}{8}\lambda^4 \\ &\qquad - i(1 + \chi + 3\lambda + \tfrac{15}{4}\lambda^2)^2 = 0 \end{aligned}\right\} \quad \text{.....(6),}$$

where $\chi = k^2, \quad j^2 = \beta_1/8, \quad i = \frac{1}{4}(\beta_2 - 3).$

The area subtended by the translated curve between any two values of x can be obtained, after solving the cubic (4), from tables of the normal probability integral. An ambiguity arises when the values of k and λ are such that for a certain range of x, the cubic has three real roots. The translated curve then loses its typical shape of rising continuously from a practically zero value to a maximum and falling at the same or at a different rate down to zero again. The singularities that occur are of two types. In Edgeworth's terminology: there is a "*break*" if $\frac{dx}{d\xi}$, the quadratic expression in the denominator of (5), becomes negative; there is a "*stop*" if the ordinate of the curve has a relative minimum value, that is to say, if $\frac{dy}{d\xi}$ has real roots other than the mode. After passing through the minimum value the curve ascends and ultimately changes abruptly from $+\infty$ to $-\infty$ at that value of x which corresponds to a root of $\frac{dx}{d\xi} = 0$. Edgeworth claimed that the method of translation is applicable especially to slightly and moderately abnormal curves; and he considered the construction as sufficiently accurate if no peculiarity occurs within a distance from the median of the translated curve corresponding to a distance of $|\xi| = 2$ from the mean of the generating curve. The tail areas cut off outside this range of about 2·83 times the standard deviation of the normal curve from its mean amount to only about 5 per-mille of the total frequency,

* Edgeworth's papers on the mathematical representation of statistical data appeared chiefly in the *Journ. Roy. Stat. Soc.* For a complete bibliography see A. L. Bowley: "F. Y. Edgeworth's Contributions to Mathematical Statistics," London, 1928, *Roy. Stat. Soc.*

and are therefore practically insignificant as compared to the central portion of the curve. They are folded over or swung round, so to speak, in the process of translation, the central portion being extended or contracted according to the nature of the data.

Now $\frac{dx}{d\xi}$ will be positive for all values of $|\xi|$ from 0 to 2, provided $k^2 < 9(\lambda + \frac{1}{12})^2$. Also, the derived function

$$\frac{1}{y} \cdot \frac{dy}{d\xi} = \frac{-2[3\lambda\xi^3 + 2k\xi^2 + (3\lambda + 1)\xi + k]}{1 + 2k\xi + 3\lambda\xi^2},$$

equated to zero, will have no real root within the region $|\xi| = 2$ other than the mode, provided $k^2 < \frac{100}{9}(\lambda + \frac{1}{15})^2$. These conditions, together with $k^2 < 3\lambda$, form a lower boundary to the χ, λ field within which the method can be applied. By assuming $\beta_2 = 15$ to be a fairly extreme case, Edgeworth obtained from the second of equations (6) an upper boundary to the χ, λ area which is to be searched for values of χ and λ, satisfying equations (6). Professor Bowley* utilised these conditions in constructing a table which shows the values of χ and λ to three decimal places for given $\beta_1/8$ and $\epsilon = \frac{1}{12}(\beta_2 - 3)$ by intervals of ·01.

The portion of the β_1, β_2 plane within which Edgeworth's hypothesis holds good, subject to the conditions laid down above, extends upwards—towards higher β's—from the broken line shown in Diagram (1). I obtained this locus by computing the values of β_1 and β_2 from equations (6) corresponding to the values of χ and λ which satisfy the lower boundary of the restricted χ, λ area. When the β's of an observed distribution lie in Pearson's Type I area below the broken line, the translated curve will present singularities within a region of $|\xi| = 2$.

To illustrate, not so much the application of the method as the nature of the singularities, I take the distribution of single births arranged according to the age of mother at birth of child (Table II, p. 153). The observed constants are:

$$\beta_1 = \cdot 100{,}603, \quad \sigma = 3{\cdot}083{,}148 \text{ (2-year unit)},$$
$$\beta_2 = 2{\cdot}430{,}327, \quad N = 631{,}682.$$

Using Diagram (1) we note that the β's fall outside the limited area; hence at least one singularity within the range $|\zeta| = |\sqrt{2}\xi| = 2{\cdot}83$ is to be expected.

The constants of the translated curve are:

$$\lambda = -\cdot 07659, \quad k = \cdot 08746, \quad a = 4{\cdot}87522,$$

and Mean-Median = ·21320.

Hence: $$y = \frac{N}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\zeta^2} \cdot \frac{1}{3{\cdot}44730\,(1 + \cdot 12369\zeta - \cdot 11488\zeta^2)},$$

where $\zeta = \sqrt{2}\xi$, the origin of the curve being at the median. The curve fitted to the observed frequencies is shown in Diagram (2).

The denominator, $\frac{dx}{d\zeta}$, becomes zero for $\zeta_1 = -2{\cdot}46076$ and $\zeta_2 = +3{\cdot}53749$,

* *Loc. cit.* pp. 123—128. The table must be entered with $\beta/8$ and not with β.

corresponding to ages of mother 18·7 and 47·1; the ordinate of the curve becomes infinite at these two points. For values of $|\zeta|$ greater than ζ_1 and ζ_2, $\frac{dx}{d\zeta}$ is negative (or, x decreases with increasing ζ) and we get the two lower (negative) branches shown in the figure. They both asymptote to the x-axis as $x \to \pm\infty$, or as $\zeta \to \mp\infty$.

The relative advantages and disadvantages of the method will be discussed more fully in a later section.

EDGEWORTH'S TRANSLATED CURVES IN RELATION TO THE PEARSON TYPES.

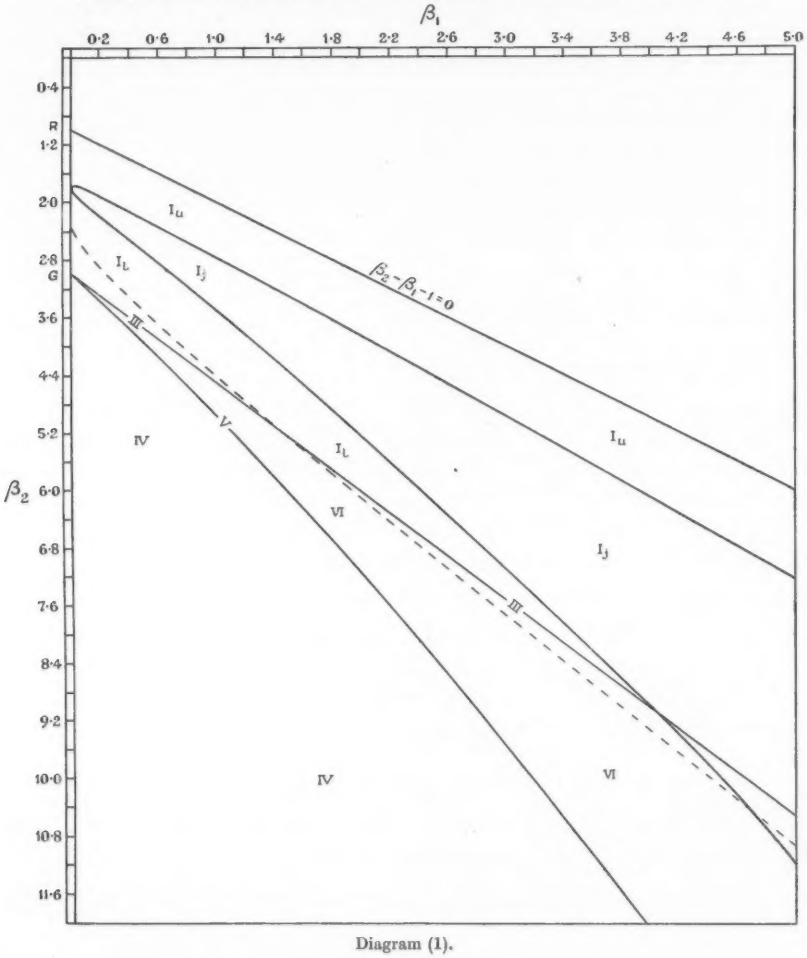

Diagram (1).

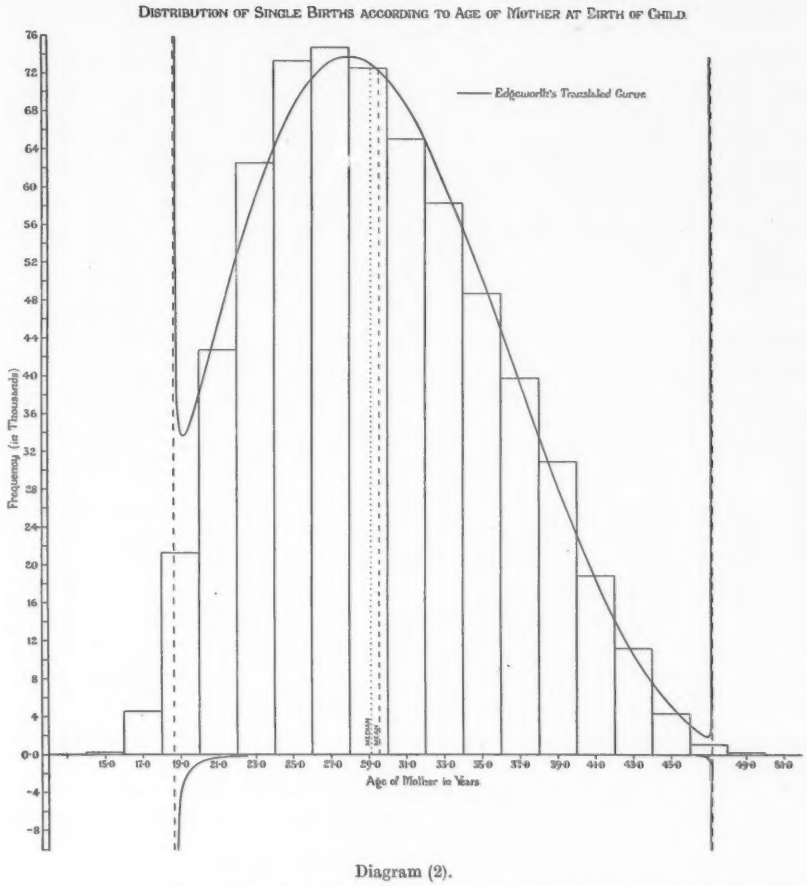
DISTRIBUTION OF SINGLE BIRTHS ACCORDING TO AGE OF MOTHER AT BIRTH OF CHILD
Edgeworth's Translated Curve
Frequency (in Thousands)
Age of Mother in Years
MEDIAN
MODE

Diagram (2).

(ii) *Logarithmic Transformation.* Galton* suggested in 1879 that in many vital phenomena the geometrical mean and not the arithmetical mean is likely to be the most probable value of the quantity measured. The corresponding law of frequency was deduced by MacAlister† in the same year. The fitting of the curve by the method of moments was discussed by Pearson‡ (1905) and more recently by Jørgensen§, Wicksell‖ and several other writers.

* "The Geometric Mean, in Vital and Social Statistics," *Proc. Roy. Soc.* Vol. XXIX. 1879, pp. 365—367.

† "The Law of the Geometric Mean," *Proc. Roy. Soc.* Vol. XXIX. 1879, pp. 367—376.

‡ "'Das Fehlergesetz und seine Verallgemeinerungen durch Fechner und Pearson.' A Rejoinder." *Biometrika*, Vol. IV. 1905—1906, pp. 193—196.

§ *Loc. cit.* pp. 47—49.

‖ "On the Genetic Theory of Frequency," *Arkiv för Mat., Astr. och Fysik*, Bd. 12, No. 20, 1917, pp. 1—56.

Consider the normal curve:

$$\eta = \frac{1}{\sqrt{2\pi}} . e^{-\frac{1}{2}\xi^2},$$

and transform it by writing:

$$\xi = \frac{\log_{10} x - l}{s} \quad \text{.....(7).}$$

The resulting frequency curve is of the form:

$$y = y_0 . \frac{1}{x} . e^{-\frac{1}{2}\left(\frac{\log x - l}{s}\right)^2} \quad \text{.....(8).}$$

The pth moment about the start of the curve is given by:

$$\begin{aligned} M_p' &= \int_0^\infty y . x^p . dx \\ &= y_0 . \sqrt{2\pi} . b . s . e^{blp + \frac{1}{2}b^2 s^2 p^2}, \end{aligned}$$

where $$\log_{10} e = 1/b.$$

For the areas under the two curves to be equal, we must have:

$$y_0 = \frac{N}{\sqrt{2\pi} . b . s}.$$

Hence: $$\mu_r' = e^{blp + \frac{1}{2}b^2 s^2 p^2}.$$

Or, taking moments about the mean:

$$\left.\begin{aligned} \mu_2 &= e^{2bl + b^2 s^2} . [e^{b^2 s^2} - 1] \\ \mu_3 &= e^{3bl + \frac{3}{2}b^2 s^2} . [e^{3b^2 s^2} - 3 . e^{b^2 s^2} + 2] \\ \mu_4 &= e^{4bl + 2b^2 s^2} . [e^{6b^2 s^2} - 4 . e^{3b^2 s^2} + 6 . e^{b^2 s^2} - 3] \end{aligned}\right\} \quad \text{.....(9).}$$

The following relations* hold between the fourth and lower order moments about the start:

$$\left.\begin{aligned} \mu_4' . (\mu_1')^2 &= (\mu_3')^2 \\ \mu_4' . (\mu_1')^5 &= (\mu_2')^3 . \mu_3' \\ (\mu_4') . (\mu_1')^8 &= (\mu_2')^6 \end{aligned}\right\} \quad \text{.....(10).}$$

If the observed distribution fixes its own start, then l and s can be determined from μ_1' and μ_2':

$$\left.\begin{aligned} bs^2 &= \log\left(\frac{\mu_2'}{\mu_1'^2}\right) \\ l &= \log\left(\frac{\mu_1'^2}{\sqrt{\mu_2'}}\right) \end{aligned}\right\} \quad \text{.....(11).}$$

In the majority of cases, however, it will be better to find the start of the curve from the moment coefficients about the mean. Let ξ_1 be the distance between the start and mean of the curve, then from (9):

$$\mu_3 \xi_1^3 - 3\mu_2^2 . \xi_1^2 - \mu_2^3 = 0 \quad \text{.....(12),}$$

* For a more complete analysis of the range of applicability of this curve, see pp. 146—150.

from which ξ_1 is to be determined. Further:

$$\left.\begin{aligned} l &= 2\log\xi_1 - \tfrac{1}{2}\log(\mu_2+\xi_1^2) \\ bs^2 &= 2\log\xi_1 - 2l \\ &= \log(\mu_2+\xi_1^2) - 2\log\xi_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13).$$

The possibility of extending this method to the Type A series, has been pointed out by Jørgensen and Wicksell in the papers cited.

3. *Schols and Perozzo.* To preserve completeness in the historical survey, as far as possible, it seems desirable before passing on to the extension of the univariate formulae to the problem of correlation, to give a brief account of Schols'* treatment (1875) of errors in space, and of Perozzo's† analysis (1882) of Italian marriage statistics.

A full theory of errors of observations in space was for the first time worked out by Schols. He dealt generally with the principal axes of inertia, and showed that for the normal surface they were axes of independent probability. Generalising this it would signify that if $z = F(x, y)$ be the expression for the frequency surface, then by a rotation of axes it could be put into the form:

$$z = f_1(x') \,.\, f_2(y').$$

Sections parallel to the principal axes are thus not only similar but also similarly situated. Any justification for applying this idea to frequency surfaces in general necessarily rests on the geometrical analysis of observed data; Schols does not seem to have attempted this.

Perozzo's investigation is, as far as I am aware, the first attempt to analyse graphically a skew bivariate distribution and to give general formulae for its representation. From the table exhibiting the number of marriages contracted in Italy during the years 1878–79 Perozzo obtains the contours of equal probability; he points out that they are not concentric and are tending to symmetry with respect to one axis only. In other words, the normal surface—which at that time was of interest only in ballistics—no longer applies. As an approximation to the binomial Perozzo gives the asymmetrical curves:

$$z = z_0 \,.\, e^{\pm a_1 x - a_2 x^2 \pm a_3 x^3 - \ldots}$$

and

$$z = a\left(x - \sqrt{\frac{n}{2a_2}}\right)^n \,.\, e^{-a_2\left(x - \sqrt{\frac{n}{2a_2}}\right)^2}.$$

Similarly for the asymmetrical surface

$$z = z_0 \,.\, z_0' \,.\, e^{-a_2 x^2 \pm a_3 x^3 - \ldots - a_2' y^2 \pm a_3' y^3 - \ldots}$$

and

$$z = a \,.\, a' \,.\, \left(x - \sqrt{\frac{n}{2a_2}}\right)^n \left(y - \sqrt{\frac{n'}{2a_2'}}\right)^{n'} \,.\, e^{-a_2\left(x - \sqrt{\frac{n}{2a_2}}\right)^2 - a_2'\left(y - \sqrt{\frac{n'}{2a_2'}}\right)^2}.$$

* "Théorie des erreurs dans le plan et dans l'espace," *Ann. de l'École Polytechn. de Delft*, 1886, pp. 123—175. Published in Dutch in the *Verhandelingen van de Koninklijke Akademie van Wetenschappen*, Deel 15, 1875, Amsterdam.

† "Nuove Applicazioni del Calcolo delle Probabilità," *Acta, Reale Accademia dei Lincei*, 1881—82, pp. 1—33.

Perozzo gives no underlying theoretical basis for these formulae, nor does he fit them to his observations.

4. *Double Hypergeometrical Series.* After having discussed the development of his system of curves (1895), Pearson remarked that if material obeyed a law of skew distribution, the Galton-Dickson theory of correlation would have to be considerably modified. The curves of equal probability derived from the correlation of cards of the same suit in two players' hands at whist, and from the correlation of ages of husband and wife at marriage, indicated a distinct deviation from the ellipses of normal correlation. The analytical description of skew bivariate distributions thus claimed immediate attention.

The idea of axes of independent probability marked the starting point of Pearson's researches on this problem. By an analysis such as that mentioned in the preceding paragraph, he was however able to convince himself that if principal inertial axes of the contour system existed, they were not axes of independent probability. The next step taken was an endeavour to extend the idea underlying his system of skew curves, i.e. to determine a family of surfaces from the two general differential equations to a certain double hypergeometrical series. These equations, as given by Rhodes*, were of the form:

$$\frac{1}{z}\cdot\frac{dz}{dx}=\frac{\text{Cubic in } x, y}{\text{Quartic in } x, y},$$

$$\frac{1}{z}\cdot\frac{dz}{dy}=\frac{\text{Another cubic in } x, y}{\text{Same quartic in } x, y}.$$

Without limitation on the constants, however, integration was found to be impossible. Special forms were thereafter considered by Pearson, Filon (1901) and Isserlis (1913), but these again led to surfaces of little value. In each case there existed a relation between the β's of the two marginal distributions, while also the correlation could be expressed as a function of them. That these and similar restrictions upon the characteristics of the distribution could not lead to satisfactory bivariate frequency surfaces, has over and over again been emphasised by Pearson. Freedom can be given to the variation of the characteristics only by having enough independent constants in the equation of the surface. The following surface is given by Pearson† as one of those obtained by Filon and Isserlis:

$$z=z_0\left(\frac{x}{b_1}\right)^{p_1}\left(\frac{y}{b_2}\right)^{p_2}\left(1-\frac{x}{b_1}-\frac{y}{b_2}\right)^{q} \qquad \text{.....(14).}$$

Here
$$\frac{\beta_{20}+3}{\beta_{10}+4}=\frac{\beta_{02}+3}{\beta_{01}+4},$$

$$r=\pm\sqrt{\frac{(p_1+1)(p_2+1)}{(p_1+q+2)(p_2+q+2)}}.$$

The marginal and array distributions are Pearson Type I curves. Regression and scedasticity are linear.

* "On a Certain Skew Correlation Surface," *Biometrika*, Vol. XIV. 1922—23, p. 355.

† "Notes on Skew Frequency Surfaces," *Biometrika*, Vol. XV. 1923, pp. 224—230.

The fitting of data with a discontinuous double hypergeometrical series was accomplished by Isserlis* in 1914. The corresponding problem in probability may be stated as follows:

Suppose a limited population of size N to contain m marked and $N-m$ unmarked characters; a sample of n is drawn and not replaced; a second sample of n' is drawn. The chance of s marked characters in the first sample and s' in the second, is

$$z(s, s') = \frac{n!\,n'!}{s!\,s'!\,(n-s)!\,(n'-s')!} \cdot \frac{(N-n-n')!}{N!} \cdot \frac{m!}{(m-s-s')!} \cdot \frac{(N-m)!}{(N-n-n'-m+s+s')!} \quad \ldots\ldots(15).$$

Let $-n=\alpha$, $-n'=\alpha'$, $-m=\beta$, $N-m-n-n'+1=\gamma$, then it can be shown that

$$\Sigma\Sigma z(s, s') = \frac{(N-n-n')!}{N!} \cdot \frac{(N-m)!}{(N-m-n-n')!} \cdot F(\alpha, \alpha', \beta, \gamma, 1, 1),$$

where $F(\alpha, \alpha', \beta, \gamma, x, y)$ denotes the double hypergeometrical series

$$\Sigma\Sigma \frac{\alpha_s \alpha'_{s'} \beta_{s+s'}}{s!\,s'!\,\gamma_{s+s'}} \cdot x^s y^{s'},$$

in which

$$a_s = a(a+1)(a+2)\ldots(a+s-1).$$

Isserlis expresses the parameters n, n', m and N in terms of moment and product-moment coefficients. He evaluates them for three numerical examples but to only one of the examples the equivalent series is fitted, namely, the distribution in 25,000 deals of trumps in the first two hands in whist with ordinary shuffling. The annexed photographs of the theoretical and observational surfaces superposed do not give us a clear idea of the goodness of fit. It is however not likely to be very good: the experimental returns show too marked discrepancies from the theoretical frequencies computed from the double hypergeometrical series†.

The range of applicability of the hypergeometrical was to some extent defined by Wicksell‡ (1917) when he showed that its regression curves are linear. In this connection it may be of interest to point out that while the discontinuous has linear regression, the two general differential equations (p. 122) lead to a surface with cubic regression§.

Wicksell‖ has further shown (1923) that the Type A and Type B series are analytical expressions for the representation of the hypergeometrical as well

* "The Application of Solid Hypergeometrical Series to Frequency Distributions in Space," *Phil. Mag.* Vol. XXVIII. 1914, pp. 379—403.

† Pearson, Karl: "On a Certain Double Hypergeometrical Series and its Representation by Continuous Frequency Surfaces," *Biometrika*, Vol. XVI. 1924, p. 186.

‡ "The Application of Solid Hypergeometrical Series to Frequency Distributions in Space," *Phil. Mag.* Vol. XXXIV. 1917, pp. 389—394.

§ Pearson, Karl: "Notes on Skew Frequency Surfaces," *Biometrika*, Vol. XV, 1923, p. 222.

‖ "Contributions to the Analytical Theory of Sampling," *Arkiv för Mat., Astr. och Fysik*, Bd. 17, No. 19, 1923.

as of the binomial. The double hypergeometrical leads to correlation functions of these two types.

In 1924 Pearson returned to the representation of a double hypergeometrical series by continuous frequency surfaces. The regression and scedasticity* are shown to be linear and parabolic respectively; a symmetrical surface† with similar forms for the regression and scedasticity is fitted to the special case of whist correlation: $N = 52$, $n = n' = 13$; also the Filon-Isserlis surface is fitted. From a comparison of the marginal distributions and of the contours, neither of the two surfaces seems to be really adequate.

5. *Skew Correlation and Non-linear Regression.* The preceding section clearly indicates that the earliest attempts at describing skew correlation, as based on the "correlation surface method," were not very profitable. Recourse had therefore to be had to a more general method which would not involve any assumptions as to the form of the frequency distribution. In a paper on multiple correlation (1897) Yule‡ showed that if the regression be linear, irrespective of the type of frequency surface, the multiple regression "plane" as reached by the method of least squares was identical in form with that flowing from a multiple normal surface. This method of approaching the problem of correlation, i.e. from the form of the regression curves, was extended by Pearson§ (1905) to non-linear regression.

Now while it is of great advantage that no assumptions as to the frequency distribution are made, this generality is, as has been pointed out by Pearson, also the chief defect of the method. Without some knowledge of the array distributions the probability of an individual observation falling within certain limits as measured from the regression curves cannot be determined.

The types of regression dealt with are: linear, parabolic, cubic and quartic. The parameters of these polynomials are expressed in terms of moments and product moments. Theoretically there is no limit to the order of the curve; in practice it depends largely on the rapidly increasing probable errors of the moments. The correlation ratio, η, is introduced as a measure of relationship when the regression is not linear; the conceptions of scedasticity and clisy are formulated, and measures of their heterogeneity are given. Finally, the regression formulae are illustrated on four examples.

A general method of determining the successive terms in a skew regression line was published by Pearson‖ in 1921. The form of the regression curve is assumed to be

$$y = f(x) = a_0\psi_0 + a_1\psi_1 + \ldots + a_n\psi_n,$$

* See pp. 141—142 for the third and fourth array moments.

† See p. 137.

‡ "On the Significance of Bravais' Formulae for Regression, etc., in the case of Skew Correlation," *Proc. Roy. Soc.* Vol. LX. 1896—97, pp. 477—489.

§ "On the General Theory of Skew Correlation and Non-Linear Regression," *Drapers' Company Research Memoirs*, Biometric Series, II. 1905, pp. 1—54.

‖ "On a General Method of determining the Successive Terms in a Skew Regression Line," *Biometrika*, Vol. XIII. 1920—21, pp. 296—300.

where $a_0, a_1, \ldots a_n$ are constants to be determined and the ψ's form an orthogonal system of functions of x. The regression orthogonal functions up to the fourth order are obtained.

The results of Pearson have been put into a still more general form by Neyman*. Certain results of the theory of continued fractions are used, but no appeal is made to their theory, nor to the theory of orthogonal functions. The nth order regression parabola is expressed in determinantal form.

6. *The Correlation Function of Type A.* The surface whose sections parallel to the coordinate planes zx and zy are curves of Type A, has been discussed by Van der Stok† (1907–1908), Charlier‡ (1914), Jørgensen§ (1916), Wicksell‖ (1917), and others on the Continent; in England by Edgeworth¶ (1896, 1905, 1917), Pearson** (1925) and Rhodes†† (1925). For brevity we shall adopt Jørgensen's notation for this surface, viz. Type AA. Its general equation can be written in the form:

$$\left.\begin{aligned} F(x,y) &= \phi(x,y) + \sum\sum_{(p+q)\geq 3} (-1)^{p+q} \cdot \frac{A_{pq}}{p!\,q!} \cdot \frac{\partial^{p+q}\phi(x,y)}{\partial x^p \cdot \partial y^q} \\ \text{where the generating function is the normal surface:} & \\ \phi(x,y) &= \frac{1}{2\pi\sqrt{1-r^2}} \cdot e^{-\frac{1}{2(1-r^2)}(x^2-2rxy+y^2)} \end{aligned}\right\} \quad \ldots\ldots\ldots(16).$$

The various contributions may be dealt with as follows: (*a*) special forms of $F(x, y)$, (*b*) determination of the coefficients of the differential terms, (*c*) the partial moment curves, (*d*) the curves of equal probability, (*e*) applications. The marginal distributions are identical with the Type A curves treated in Section 2.

(*a*) *Special forms of* $F(x, y)$. With the exception of Edgeworth and Van der Stok, all the authors mentioned above start with equation (16). The ensuing discussions of Charlier, Jørgensen and Pearson are confined to the approximation $3 \leqslant (p+q) \leqslant 4$ (Type AaAa); Wicksell discusses both this approximation and that given in Section 2 where all terms of the order $1/n$ are included (Type AbAb); Edgeworth extends his generalised law of error to two dimensions but considers thereafter terms involving moments up to the third order only, $(p+q)=3$; this approximation is discussed more fully by Rhodes who applies it to the problem of ranks and grades.

* "Further Notes on Non-Linear Regression," *Biometrika*, Vol. XVIII. 1926, pp. 257—262.

† "On the Analysis of Frequency Curves according to a General Method," *Proc. Kon. Ak. v. Wet.* (Amsterdam), 1907—1908, pp. 799—817.

‡ "Contributions to the Mathematical Theory of Statistics. 6. The Correlation Function of Type A," *Arkiv för Mat., Astr. och Fysik*, Bd. 9, No. 26, 1914, pp. 1—18.

§ *Undersøgelser over Frequensflader og Korrelation.* København, 1916: Arnold Busck.

‖ "The Correlation Function of Type A, and the Regression of its Characteristics," *Kungl. Sv. Vet. Akad. Handl.* Bd. 58, No. 3, 1917, pp. 1—48.

¶ "The Compound Law of Error," *Phil. Mag.* Vol. XLI. 1896, pp. 207—215; "The Law of Error," *Camb. Phil. Trans.* Vol. XX. 1905, pp. 116—119; "On the Mathematical Representation of Statistical Data," *Journ. Roy. Stat. Soc.* Vol. LXXX. 1917, pp. 266—288.

** "The Fifteen Constant Bivariate Frequency Surface," *Biometrika*, Vol. XVII. 1925, pp. 268—313.

†† "On a Skew Correlation Surface," *Biometrika*, Vol. XVII. 1925, pp. 314—326.

Van der Stok takes as generating function:

$$\phi_1(x, y) = \frac{1}{2\pi} \cdot e^{-\frac{1}{2}(x^2+y^2)} = f_1(x) \times f_2(y) \ldots\ldots\ldots\ldots\ldots\ldots (17),$$

and deduces the surface:

$$F_1(x, y) = \phi_1(x, y) + B_{11} \frac{df_1}{dx} \cdot \frac{df_2}{dy} + \underset{(p+q)\geq 3}{\Sigma\Sigma} (-1)^{p+q} \frac{B_{pq}}{p!\,q!} \cdot \frac{d^p f_1}{dx^p} \cdot \frac{d^q f_2}{dy^q} \ldots (18).$$

The only comment he makes on this surface is that by a rotation of axes the B_{11}-term can be made to vanish.

Jørgensen observes that the general form (16) is not well adapted to numerical application; he thereupon turns to (17) and (18) where the variables and differential coefficients are separable. With tables of the normal curve and of its derivatives at hand, the arithmetical work can be greatly diminished.

The fitting of equation (16) can be best performed after the differential functions have been expanded as a polynomial in x and y. In this form the surface is discussed by Pearson:

$$F(x, y) = \phi(x, y) \cdot [1 - a_0 + a_1 x + a_2 y + b_1 x^2 + 2b_2 xy + b_3 y^2 + c_1 x^3 + c_2 x^2 y + c_3 xy^2 + c_4 y^3 + d_1 x^4 + d_2 x^3 y + 3d_3 x^2 y^2 + d_4 xy^3 + d_5 y^4] \ldots (19).$$

(*b*) *The coefficients* in the different equations, as given by the respective writers, are as follows:

Type AaAa.

$$\left.\begin{array}{ll} A_{30} = q_{30}, & A_{40} = q_{40} - 3 \\ A_{21} = q_{21}, & A_{31} = q_{31} - 3r \\ A_{12} = q_{12}, & A_{13} = q_{13} - 3r \\ A_{03} = q_{03}, & A_{04} = q_{04} - 3 \\ A_{22} = q_{22} - 1 - 2r^2 & \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (20).$$

Type AbAb. The coefficients of the additional terms ($p + q = 6$) are:

$$\left.\begin{array}{ll} A'_{60} = \frac{1}{2}(A'_{30})^2, & A'_{06} = \frac{1}{2}(A'_{03})^2 \\ A'_{51} = A'_{30} \cdot A'_{21}, & A'_{15} = A'_{03} \cdot A'_{12} \\ A'_{42} = A'_{30} \cdot A'_{12} + \frac{1}{2}(A'_{21})^2, & A'_{24} = A'_{03} \cdot A'_{21} + \frac{1}{2}(A'_{12})^2 \\ A'_{33} = A'_{30} \cdot A'_{03} + A'_{21} \cdot A'_{12} & \end{array}\right\} \ldots\ldots (21),$$

where

$$A'_{pq} = (-1)^{p+q} \cdot \frac{A_{pq}}{p!\,q!}.$$

Equation (18). Except for B_{22}, the B-coefficients are identical with the A's:

$$B_{11} = q_{11} = r, \quad B_{22} = q_{22} - 1 \ldots\ldots\ldots\ldots\ldots\ldots (22).$$

Equation (19). Pearson gives the expressions for the coefficients a, b, c and d in the paper already cited. Because of their complexity I shall not re-write them.

(*c*) *The Partial Moment Curves.* The regression and scedastic curves of Type AaAa are derived by Wicksell and Pearson. The third and fourth partial moments are found only by Wicksell; he hereupon treats the special forms all these moment curves will assume when the correlation is moderately skew.

Pearson's forms are:

Regression Curve of y on x:

$$\mu_1'(y) = rx + \frac{\sqrt{\frac{5}{2}}(q_{21} - r\sqrt{\beta_{10}})\tau_3 + \sqrt{\frac{5}{3}}(q_{31} - r\beta_{20})\tau_4}{\tau_1 + \sqrt{\frac{9}{5}} \cdot \sqrt{\beta_{10}} \cdot \tau_4 + \sqrt{\frac{5}{24}} \cdot \overline{\beta_{20} - 3} \cdot \tau_5} \quad \ldots\ldots\ldots\ldots(23)$$

$$\equiv rx + \frac{A_x}{z_x}.$$

Scedastic Curve of y on x:

$$\sigma^2(y) = 1 - r^2$$

$$- \frac{\sqrt{2}\,[r(q_{21} - r\sqrt{\beta_{10}}) - (q_{12} - rq_{21})]\tau_2 + \sqrt{6}\,[r(q_{31} - r\beta_{20}) - \frac{1}{2}(q_{22} - 1 - r^2 \cdot \overline{\beta_{20} - 1})]\tau_3}{z_x}$$

$$- \left(\frac{A_x}{z_x}\right)^2 \quad \ldots\ldots\ldots\ldots(24)$$

$$\equiv 1 - r^2 - \left(\frac{B_x}{z_x}\right) - \left(\frac{A_x}{z_x}\right)^2.$$

The following form in which Wicksell writes these curves is certainly not as elegant as (23) and (24); the deviation from normality is obscured by not taking out the factors rx and $\overline{1 - r^2}$:

$$\mu_1'(y) = \frac{rx - rA'_{30} \cdot R_4(x) - A'_{21} \cdot R_2(x) - rA'_{40} R_5(x) - A'_{31} R_3(x)}{1 + A'_{30} R_3(x) + A'_{40} \cdot R_4(x)} \quad \ldots(25),$$

where $R_s(x)$ is the Hermite Polynomial of the sth order. Wicksell develops the denominator of (25) as a power series in $A'_{30} R_3(x) + A'_{40} R_4(x)$, and neglects all terms whose coefficients are of an order less than $\frac{1}{n}$, n being the number of elementary "error-sources." Arranging the resulting expression in powers of x, he finds the regression to be cubic. To the same degree of approximation the scedasticity is parabolic, the clisy linear and the kurtosis constant. The foregoing development is justifiable only for distributions of moderate skewness and within certain ranges from the mean.

For Type AbAb Wicksell derives only the regression and scedastic curves.

Jørgensen derives the regression and scedastic curves for the simplified form (18).

(*d*) *Curves of Equal Probability.* In any correlation distribution, $F(x, y)$, the curves of equal probability are given by*

$$z = F(x, y) = \text{constant} \quad \ldots\ldots\ldots\ldots(26).$$

But more than often the form of $F(x, y)$ is too complicated for these curves to be directly constructed.

For the Type AaAa surface an approximate solution to (26) has been found by Wicksell†. Assuming the correlation to be moderately skew, he shows that in the

* The curves so defined are, strictly speaking, curves of equal ordinates, i.e. the contours of the surface.

† "The Construction of the Curves of Equal Frequency in case of Type A Correlation," *Sv. Akt. Tidskr.* Häft. 2—3, 1917, pp. 1—19.

vicinity of the mode the curves of equal probability are ellipses, while further out they are disturbed ellipses. These outer "ellipses" can be constructed by making use of auxiliary circles; certain quantities expressed in terms of products of the Hermite Polynomials are to be added to the radii of the circles for the radii vectores of the required curves to be obtained. Tables are given for $R_i(\xi_1) . R_j(\eta_1)$, where ξ_1 and η_1 are the points of intersection of 24 radii vectores with circles whose radii correspond to definite values of $z = F(x, y) =$ constant.

A more detailed treatment or a restatement of the derived formulae seems to me not warranted.

(*e*) *Applications.* The fitting of surface (18) is illustrated by Jørgensen on one example. He first considers the possibility of making some of the higher coefficients in the expression negligibly small by a rotation of axes; the new axes are to coincide with the principal axes of inertia. However, in his particular illustration nothing is gained by such a transformation. The mid-ordinates of the frequency cells are calculated and compared with the observed frequencies. Even if allowance be made for the paucity of the observations we are bound to conclude, from an examination of the table, that the graduation is not at all satisfactory.

Wicksell illustrates his method of approximating to the partial moment curves of the Type AaAa surface on four examples representative of moderately and of considerably skew correlation. Both regression curves are fitted for all four examples; the scedastic curves for two of the examples only. The range of applicability of the approximate formulae can to some extent be appreciated from the following values of β_1 and β_2 · ich I have evaluated for the marginal distributions corresponding to the instances where Wicksell replaces his cubic by the general regression curve (25):

$$\left.\begin{matrix}\beta_1 = \cdot 109 \\ \beta_2 = 2{\cdot}952\end{matrix}\right\}; \quad \left.\begin{matrix}\beta_1 = \cdot 233 \\ \beta_2 = 2{\cdot}889\end{matrix}\right\}; \quad \left.\begin{matrix}\beta_1 = \cdot 829 \\ \beta_2 = 4{\cdot}863\end{matrix}\right\}.$$

Hereafter, Wicksell fits his formulae to three of the examples given by Pearson in his memoir on skew correlation and non-linear regression. The diagrams given by Wicksell seem to indicate that his formulae, with moments up to the fourth order, give virtually as good a description of the observation points as Pearson's formulae involving moments up to the sixth; also the arithmetic is far less. However, not until we have more comparative results before us, will it be possible to vindicate the general use of these formulae.

Pearson tests the value of "The Fifteen Constant Surface" (19) on two examples: the whist double hypergeometrical series, and the distribution of contemporaneous barometric heights at Southampton and Laudale. In both illustrations the theoretical ordinates and frequencies are computed, and the contour lines are constructed. A very close agreement is obtained between the mathematical surface and the double hypergeometrical. For the barometric data, due regard being paid to the sparseness of the observations, the agreement is less satisfactory; the Goodness of Fit Test shows, however, that the graduation is better than that

obtained by Rhodes with his surface*; the regression curves fit the observation points very well.

7. *The Correlation Functions of Type B, and of Type A and Type B.* In his systematic treatise on frequency surfaces and correlation, 1916, Jørgensen discusses the following three types of surfaces: (i) Type AA; (ii) Type BB, where sections parallel to the coordinate planes zx and zy are curves of Type B; (iii) Type AB where sections parallel to the plane zx are curves of Type A and sections parallel to the plane zy are curves of Type B.

Jørgensen takes the generating function of Type BB and of Type AB to be

$$\Im(x, y) = \Im(x) \times \Im(y)$$

and

$$\psi(x, y) = \phi(x) \times \Im(y)$$

respectively, where $\phi(x)$ and $\Im(x)$ are as defined in Section 2, p. 115. The constants, regression and scedastic curves are determined for these simplified forms. No numerical illustrations are given.

8. *Translation applied to Correlation. Edgeworth*†.

(*a*) *Simple Translation.* Let the generating surface be

$$\zeta = \frac{1}{\pi\sqrt{1-R^2}} \cdot e^{-\frac{1}{(1-R^2)}[\xi^2 - 2R\xi\eta + \eta^2]} \quad \text{.....(27),}$$

and the equations of translation:

$$\left.\begin{aligned} x &= a_1(\xi + k_1\xi^2 + \lambda_1\xi^3) \\ y &= a_2(\eta + k_2\eta^2 + \lambda_2\eta^3) \end{aligned}\right\} \quad \text{.....(28).}$$

The constants are to be determined separately for the two equations.

Taking r to be the correlation coefficient between x and y, and R to be that between ξ and η, Edgeworth finds

$$r\sqrt{\mu_{20} \cdot \mu_{02}} = a_1 \cdot a_2 \left[\frac{R}{2} + \tfrac{3}{4}(\lambda_1 + \lambda_2)R + \tfrac{1}{2}k_1 \cdot k_2 R^2 + \tfrac{1}{8}\lambda_1\lambda_2(9R + 6R^3)\right].$$

If cubic terms in k and λ are neglected, then:

$$R = r[1 + \tfrac{3}{4}(\lambda_1^2 + \lambda_2^2) + \tfrac{1}{2}(k_1^2 + k_2^2)] - k_1k_2r^2 - \tfrac{3}{2}\lambda_1\lambda_2 \cdot r^3.$$

After ξ and η have been evaluated from (28) the cell frequencies can be found from (27) with the use of tables for the normal curve.

(*b*) *Composite Translation.* Professor Bowley‡ considers the case

$$x = a_1(\xi + k_1\xi^2 + \lambda_1\xi^3 + \gamma_1\eta^2),$$
$$y = a_2(\eta + k_2\eta^2 + \lambda_2\eta^3 + \gamma_2\xi^2),$$

while Edgeworth omits λ_1 and λ_2.

* See pp. 134—136.

† "On the Use of Analytical Geometry to represent Certain Kinds of Statistics," *Journ. Roy. Stat. Soc.* Vol. LXXVII. 1914, pp. 838—852; Vol. LXXX. 1917, pp. 266—288.

‡ *F. Y. Edgeworth's Contributions to Mathematical Statistics*, Roy. Stat. Soc., London, 1928, pp. 79—81.

Fairly simple expressions can be found for k_1, k_2, γ_1 and γ_2 if squared terms in the k, λ and γ are neglected, i.e. if the correlation be regarded as *moderately* skew. To solve the general moment equations involving these constants would be a severe task.

Composite translation is necessary if the relations

$$\left.\begin{aligned} q_{21} &= \tfrac{1}{3}(2q_{30} \,.\, r + q_{03} \,.\, r^2) \\ q_{12} &= \tfrac{1}{3}(2q_{03} \,.\, r + q_{30} \,.\, r^2) \end{aligned}\right\} \quad \text{.....(29)}$$

do not hold.

Edgeworth states his views on the relative merits of simple and composite translation and of the generalised law of error in the concluding paragraph of his paper in *J. S. S.*, Vol. LXXX, 1917: "The inadequacy of simple translation, the impracticability of composite translation, constitutes an important point in the comparison between the use of the generalised law of error and the method of translation in two dimensions. The balance between the two methods is altered in one respect. Whereas in one dimension the generalised law is theoretically at least preferable for subnormal curves, while translation has the advantage of being applicable to abnormal cases, this advantage is greatly reduced in two dimensions, while that preference still subsists."

The results of both methods are illustrated on a few frequency groups. The agreement between theory and observation seems to be quite satisfactory; but whether the same degree of agreement holds throughout the surface, Edgeworth did not establish.

9. *Logarithmic Correlation. Wicksell.* The method of logarithmic transformation has been extended to correlation problems by Wicksell* (1917) in two successive papers. If $\log x$ and $\log y$ are assumed to be normally distributed, their correlation function will be

$$F(x, y) = \frac{Z_0}{x \,.\, y} \,.\, e^{-\frac{1}{2}\frac{1}{1-\rho^2}\left[\left(\frac{\log x - l_1}{s_1}\right)^2 - 2\rho\left(\frac{\log x - l_1}{s_1}\right)\left(\frac{\log y - l_2}{s_2}\right) + \left(\frac{\log y - l_2}{s_2}\right)^2\right]}.$$

The regression curves of this surface are however of a form one would not expect to observe in practice; they have (i) no inflexions, (ii) two points of intersection.

In the second paper Wicksell assumes the distribution to be of the form

$$F(\xi, \eta) = \phi(\xi, \eta) + \sum\sum_{(p+q)=3} B_{pq} \frac{\partial^{p+q} \phi(\xi, \eta)}{\partial \xi^p \,.\, \partial \eta^q} \quad \text{.....(30),}$$

where $\xi = \log x$, $\eta = \log y$, and

$$\phi(\xi, \eta) = \frac{N(\log e)^2}{s_1 \,.\, s_2 \,.\, 2\pi\sqrt{1-\rho^2}} \,.\, e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{\xi - l_1}{s_1}\right)^2 - 2\rho\left(\frac{\xi - l_1}{s_1}\right)\left(\frac{\eta - l_2}{s_2}\right) + \left(\frac{\eta - l_2}{s_2}\right)^2\right]} \quad \text{(31).}$$

* "On the Genetic Theory of Frequency," *Arkiv för Mat., Astr. och Fysik*, Bd. 12, No. 20, 1917: "On Logarithmic Correlation, with an Application to the Distribution of Ages at First Marriage," *Sv. Akt. Tidskr.* Häft. 4, 1917, pp. 1—21.

(*a*) *Determination of the constants in equation* (30). The moments, M'_{pq}, of x and y about the origin are given by:

$$M'_{pq}=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{pb\xi}\,.\,e^{qb\eta}\,.\,F(\xi,\,\eta)\,d\xi\,.\,d\eta,$$

where $b=\dfrac{1}{\log_{10} e}$ as before.

Let the origin $(\xi_1,\,\eta_1)$ be so chosen that B_{21} and B_{03} vanish; the expressions for the moments of the marginal distributions are then identical with those given on p. 120. The expressions for the product moments about the origin are:

$$\left.\begin{aligned}
\mu'_{11}&=[1-b^3(B_{21}+B_{12})]\,.\,e^{bl_1+bl_2+\frac{b^2}{2}(s_1^2+2\rho s_1\,.\,s_2+s_2^2)}\\
\mu'_{21}&=[1-2b^3(2B_{21}+B_{12})]\,.\,e^{2bl_1+bl_2+\frac{b^2}{2}(4s_1^2+4\rho s_1\,.\,s_2+s_2^2)}\\
\mu'_{12}&=[1-2b^3(B_{21}+2B_{12})]\,.\,e^{bl_1+2bl_2+\frac{b^2}{2}(s_1^2+4\rho s_1\,.\,s_2+4s_2^2)}\\
\mu'_{31}&=[1-3b^3(3B_{21}+B_{12})]\,.\,e^{3bl_1+bl_2+\frac{b^2}{2}(9s_1^2+6\rho s_1\,.\,s_2+s_2^2)}\\
\mu'_{22}&=[1-8b^3(B_{21}+B_{12})]\,.\,e^{2bl_1+2bl_2+\frac{b^2}{2}(4s_1^2+8\rho s_1\,.\,s_2+4s_2^2)}\\
\mu'_{13}&=[1-3b^3(B_{21}+3B_{12})]\,.\,e^{bl_1+3bl_2+\frac{b^2}{2}(s_1^2+6\rho s_1\,.\,s_2+9s_2^2)}
\end{aligned}\right\}\quad\ldots(32).$$

The origin $(\xi_1,\,\eta_1)$ is to be determined from

$$\left.\begin{aligned}
\mu_{30}\xi_1^3-3\mu_{20}\xi_1^2-\mu_{20}^3=0\\
\mu_{03}\eta_1^3-3\mu_{02}\eta_1^2-\mu_{02}^3=0
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(33).$$

Further

$$\left.\begin{aligned}
l_1&=2\log\xi_1-\tfrac{1}{2}\log(\mu_{20}+\xi_1^2)\\
l_2&=2\log\eta_1-\tfrac{1}{2}\log(\mu_{02}+\eta_1^2)\\
bs_1^2&=\log(\mu_{20}+\xi_1^2)-2\log\xi_1\\
bs_2^2&=\log(\mu_{02}+\eta_1^2)-2\log\eta_1
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\ldots\ldots(34).$$

Write

$$\left.\begin{aligned}
k'_{11}&=\frac{\mu'_{11}}{\mu'_{10}\,.\,\mu'_{01}}=[1-b^3(B_{21}+B_{12})]\,.\,e^{\rho s_1 s_2 b^2}\\
k'_{21}&=\frac{\mu'_{21}}{\mu'_{20}\,.\,\mu'_{01}}=[1-2b^3(2B_{21}+B_{12})]\,.\,e^{2\rho s_1 s_2 b^2}\\
k'_{12}&=\frac{\mu'_{12}}{\mu'_{02}\,.\,\mu'_{10}}=[1-2b^3(B_{21}+2B_{12})]\,.\,e^{2\rho s_1 s_2 b^2}
\end{aligned}\right\}\quad\ldots\ldots\ldots(35),$$

and assume $(b^3B_{21})^2$ and $(b^3B_{12})^2$ to be negligibly small as compared with 1. B_{12}, B_{21} and ρ are then to be found from

$$\left.\begin{aligned}
k'_{21}-k'^2_{11}&=u(2k'^2_{11}-k'_{21})+v(k'^2_{11}-k'_{21})\\
k'_{12}-k'^2_{11}&=u(k'^2_{11}-k'_{12})+v(2k'^2_{11}-k'_{12})\\
\rho&=\frac{\log k'_{11}-\log(1+u/2+v/2)}{b\,.\,s_1\,.\,s_2}
\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots(36),$$

where $u=-2b^3B_{21}$ and $v=-2b^3B_{12}$.

The successful application of these formulae depends on the following conditions: (i) that terms of an order higher than the third in (30) may always be neglected when the origin is so chosen that B_{03} and B_{30} vanish; (ii) that $(b^3B_{21})^2$ and $(b^3B_{12})^2$ are negligibly small. If this condition be not fulfilled, equations (35) must be solved for B_{21}, B_{12} and ρ. Thus

$$e^{\rho s_1 s_2 b^2} = \tfrac{3}{4}k'_{11} + \sqrt{\tfrac{1}{16}(9k'^2_{11} - 4k'_{21} - 4k'_{12})} \quad \text{.....(37)}$$

and two linear equations involving B_{12} and B_{21}.

The following identical relations between moments of the fourth order, about the origin, must be approximately fulfilled:

$$\left.\begin{aligned}
\mu'_{40}\,.\,(\mu'_{10})^8 &= (\mu'_{20})^6 \\
\mu'_{31}\,.\,(\mu'_{10})^6\,.\,(\mu'_{01})^2 &= (\mu'_{11})^3\,.\,(\mu'_{20})^3\;\frac{1-3b^3(3B_{21}+B_{12})}{[1-b^3(B_{21}+B_{12})]^3} \\
\mu'_{22}\,.\,(\mu'_{10})^4\,.\,(\mu'_{01})^4 &= (\mu'_{11})^4\,.\,\mu'_{20}\,.\,\mu'_{02}\,\frac{1-8b^3(B_{21}+B_{12})}{[1-b^3(B_{21}+B_{12})]^4} \\
\mu'_{13}\,.\,(\mu'_{10})^2\,.\,(\mu'_{01})^6 &= (\mu'_{11})^3\,.\,(\mu'_{02})^3\,\frac{1-3b^3(B_{21}+3B_{12})}{[1-b^3(B_{21}+B_{12})]^3} \\
\mu'_{04}\,.\,(\mu'_{01})^8 &= (\mu'_{02})^6
\end{aligned}\right\} \quad \text{.....(38).}$$

(*b*) *The Marginal Distributions of the Surface.* These are identical with the curves considered in Section C, 2, p. 120.

(*c*) *The Partial Moment Curves.* Wicksell finds the expression for the sth moment curve of y on x about the origin (ξ_1, η_1) to be

$$\mu_s'(y) = \mu'_{0s}\,.\,e^{\lambda^{(s)}.\log x - \gamma^{(s)}}.[D_0^{(s)} + D_1^{(s)}\{\log x - d^{(s)}\} + D_2^{(s)}\{\log x - d^{(s)}\}^2] \quad \text{.....(39),}$$

where

$$\left.\begin{aligned}
\lambda^{(s)} &= s\left(b\rho\,.\,\frac{s_2}{s_1}\right), & D_2^{(s)} &= -s^2\left(b\,.\,\frac{B_{21}}{s_1^4}\right) \\
D_0^{(s)} &= 1 + s\left(b\,.\,\frac{B_{21}}{s_1^2}\right), & \gamma^{(s)} &= s\left(b\rho\,.\,\frac{s_2}{s_1}\,.\,l_1\right) + s^2(\tfrac{1}{2}b^2s_2^2\rho^2) \\
D_1^{(s)} &= -s^2\left(b^2\,.\,\frac{B_{12}}{s_1^2}\right), & d^{(s)} &= l_1 + s(bs_1s_2\rho)
\end{aligned}\right\} \quad \text{.....(40).}$$

(*d*) *Illustration.* The derived formulae are fitted to the marginal and regression curves for the age distribution of bachelors and spinsters married in Sweden, 1901—10. The relative marginal frequencies and a diagram of the regression curves seem to indicate a fair agreement between theory and observation.

10. *Steffensen's Correlation Formulae**. To represent a slight degree of correlation, Steffensen (1922) writes the frequency function in the form:

$$F(x, y) = kf_1(x, y) \times f_2(x, y),$$

which, for special values of the parameters, can be reduced to

$$F(x, y) = k_1 f_1'(x) \times f_2'(y).$$

* 'A Correlation Formula,' *Sk. Akt. Tidskr.* 1922; *Matematisk Iagttagelselære*, København, 1923, pp. 106—132.

Suppose x and y to be linearly related, then

$$F(x, y) = kf_1(x + cy) \times f_2(y + \gamma x) \quad \ldots\ldots\ldots\ldots\ldots\ldots (41)$$
$$= kf_1(\xi) \times f_2(\eta),$$

where
$$\left.\begin{aligned} \xi &= x + cy \\ \eta &= y + \gamma x \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (42).$$

Determination of the Constants in (41). Let c_p' and γ_p' denote the moment coefficients of the functions $f_1(\xi)$ and $f_2(\eta)$ respectively; μ'_{pq} the pqth moment coefficient of $F(x, y)$; the dashes are to be dropped when the origin is at the mean. Thus

$$c_p' = \int \xi^p f_1(\xi)\, d\xi, \quad \gamma_p' = \int \eta^p f_2(\eta)\, d\eta,$$

$$\mu'_{pq} = \iint x^p . y^q . F(x, y)\, dx\, dy.$$

The constants on which f_1 and f_2 depend are to be found in the usual way from the moments c_p and γ_p.

From transformation (42):

$$\iint f_1(\xi) f_2(\eta)\, d\xi . d\eta = \iint f_1(x + cy) . f_2(y + \gamma x) \begin{vmatrix} \frac{\partial \xi}{\partial x}, & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y}, & \frac{\partial \eta}{\partial y} \end{vmatrix} dx\, dy$$
$$= \frac{1}{k} \iint F(x, y) . |1 - c\gamma| . dx\, dy,$$

i.e.
$$k = |1 - c\gamma| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (43).$$

Also
$$\iint (x + cy)^p . (y + \gamma x)^q . F(x, y)\, dx\, dy = c_p' . \gamma_q' \quad \ldots\ldots\ldots\ldots (44).$$

If the origin is assumed to be at the mean, it follows from (44) that

$$c_1 = \mu_{10} + c\mu_{01} = 0,$$
$$\gamma_1 = \gamma\mu_{10} + \mu_{01} = 0,$$
$$\gamma\mu_{20} + (1 + c\gamma)\mu_{11} + c\mu_{02} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (45),$$

$$\left.\begin{aligned} \gamma^2\mu_{30} + \gamma(2 + c\gamma)\mu_{21} + (1 + 2c\gamma)\mu_{12} + c\mu_{03} &= 0 \\ \gamma\mu_{30} + (1 + 2c\gamma)\mu_{21} + c(2 + c\gamma)\mu_{12} + c^2\mu_{03} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (46).$$

Equations (45) and (46) are to be solved for c and γ.

Write
$$u = \frac{\gamma}{1 + c\gamma}, \quad v = \frac{c}{1 + c\gamma},$$

whence
$$c = \frac{1 - \sqrt{1 - 4uv}}{2u}, \quad \gamma = \frac{1 - \sqrt{1 - 4uv}}{2v}.$$

Substitute these expressions for c and γ in (45) and (46):

$$u\mu_{20} + v\mu_{02} + \mu_{11} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (47),$$

$$\left.\begin{aligned} u\mu_{21} + v\mu_{03} + \mu_{12} &= 0 \\ u\mu_{30} + v\mu_{12} + \mu_{21} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (48).$$

Combine equations (48) by making

$$(u\mu_{21}+v\mu_{03}+\mu_{12})^2+(u\mu_{30}+v\mu_{12}+\mu_{21})^2$$

a minimum. This gives

$$\frac{\mu_{02}}{\mu_{20}}=\frac{u(\mu_{03}\cdot\mu_{21}+\mu_{30}\cdot\mu_{12})+v(\mu_{03}^2+\mu_{12}^2)+\mu_{12}(\mu_{03}+\mu_{21})}{u(\mu_{30}^2+\mu_{21}^2)+v(\mu_{03}\cdot\mu_{21}+\mu_{30}\cdot\mu_{12})+\mu_{21}(\mu_{30}+\mu_{12})} \quad \text{.....(49).}$$

From (47) and (49) c and γ can now be easily determined.

The moments c_p and γ_p are obtained by expanding the binomial in (44). Thus

$$\left.\begin{aligned} c_p &= \mu_{p0}+\frac{p}{1!}\cdot c\cdot\mu_{p-1,1}+\frac{p(p-1)}{2!}\cdot c^2\cdot\mu_{p-2,2}+\ldots+c^p\mu_{0p} \\ \gamma_p &= \gamma^p\mu_{p0}+\frac{p}{1!}\cdot\gamma^{p-1}\cdot\mu_{p-1,1}+\frac{p(p-1)}{2!}\cdot\gamma^{p-2}\cdot\mu_{p-2,2}+\ldots+\mu_{0p} \end{aligned}\right\} \text{.....(50).}$$

Application. The method is illustrated on the example treated by Jørgensen for the simplified form of Type AaAa, equation (18). The moments c_p and γ_p are evaluated; β_1 and β_2 for each of the functions f_1 and f_2 correspond approximately to a Pearson Type III curve. The resulting equation of the surface is of the form:

$$z=z_0\cdot e^{-d_1x-d_2y}(1-a_1x+b_1y)^{p_1}(1-a_2x+b_2y)^{p_2} \text{.....(51).}$$

The cell mid-ordinates are computed and exhibited together with Jørgensen's result. From an inspection of the table it is fairly obvious that Steffensen's method gives the better graduation. Moreover, it does not give rise to the objectionable negative frequencies.

11. *Rhodes' Surface** (1922). The equation† of the surface is

$$z=z_0\cdot e^{-lx-my}\left(1-\frac{x}{a}+\frac{y}{b}\right)^p\left(1+\frac{x}{a'}-\frac{y}{b'}\right)^{p'} \text{.....(52),}$$

the mode being given by:

$$1-\frac{x}{a}+\frac{y}{b}=\frac{pp'\left(\frac{1}{ab'}-\frac{1}{a'b}\right)}{-\left(\frac{mp'}{a'}+\frac{lp'}{b'}\right)},$$

$$1+\frac{x}{a'}-\frac{y}{b'}=\frac{pp'\left(\frac{1}{ab'}-\frac{1}{a'b}\right)}{-\left(\frac{lp}{b}+\frac{mp}{a}\right)}.$$

(*a*) *Determination of the Constants in* (52). By considering integrals of the form $\iint\frac{dz}{dx}\cdot x^t\cdot y^{t'}\cdot dxdy$ and $\iint\frac{dz}{dy}\cdot x^t\cdot y^{t'}\cdot dxdy$, Rhodes finds the following equations for the determination of θ, ϕ, λ, s and z_0:

$$\beta_{10}=\frac{4(\phi^3+\lambda)^2}{s(\phi^2+\lambda)^3}, \qquad \beta_{01}=\frac{4(\theta^3+\lambda)^2}{s(\theta^2+\lambda)^3},$$

* "On a Certain Skew Correlation Surface," *Biometrika*, Vol. XIV. 1922—23, pp. 355—377.

† Equations (51) and (52) are of essentially the same form. Rhodes' Surface can be obtained from the product of two Pearson Type III curves by linear transformations of the arguments.

$$r=\frac{\theta\phi+\lambda}{\sqrt{(\theta^2+\lambda)(\phi^2+\lambda)}}, \qquad z_0=\frac{N.X.p^s.p'^{s'}}{e^{R-2}.\Gamma(s).\Gamma(s')},$$

$$\left.\begin{aligned}\frac{q_{21}-r\sqrt{\beta_{10}}}{\sqrt{1-r^2}}&=\frac{2\sqrt{\lambda}}{\sqrt{s}}\cdot\frac{\phi(1-\phi)}{(\phi^2+\lambda)^{\frac{3}{2}}}\\ \frac{q_{12}-r\sqrt{\beta_{01}}}{\sqrt{1-r^2}}&=\frac{2\sqrt{\lambda}}{\sqrt{s}}\cdot\frac{\theta(1-\theta)}{(\theta^2+\lambda)^{\frac{3}{2}}}\end{aligned}\right\}\quad\text{.....(53),}$$

where

$$\theta=\frac{ap'}{a'p}, \quad \phi=\frac{bp'}{b'p}, \quad \lambda=\frac{s'}{s},$$

$$X=\frac{1}{a'b}-\frac{1}{ab'}, \quad R=p+p'+2, \quad s'=p'+1, \quad s=p+1.$$

In the illustration one equation is formed from the two equations (53) so as not to give greater weight to one part of the table; but Rhodes does not tell us how he combines them.

The following relations hold amongst the moments of the surface:

$$\left.\begin{aligned}\frac{q_{21}-r\sqrt{\beta_{10}}}{\sqrt{1-r^2}}.\sqrt{3}&=\sqrt{2\beta_{20}-3\beta_{10}-6}\\ \frac{q_{12}-r\sqrt{\beta_{01}}}{\sqrt{1-r^2}}.\sqrt{3}&=\sqrt{2\beta_{02}-3\beta_{01}-6}\end{aligned}\right\}\quad\text{.....(54).}$$

The distance of the mean from the mode is

$$\mu'_{10}=\frac{1}{X}\left(\frac{1}{pb'}+\frac{1}{p'b}\right),$$

$$\mu'_{01}=\frac{1}{X}\left(\frac{1}{pa'}+\frac{1}{p'a}\right).$$

(*b*) *The Arrays of the Surface.* The marginal and regression curves are expressed as infinite series:

y-marginal curve:

$$z_y=c.e^{-pa'u}.\left[u^{R-1}-\frac{aa'ls'}{R}.u^R+\frac{(aa'l)^2}{2!}\cdot\frac{s'(s'+1)}{R.R+1}.u^{R+1}-\ldots\right].$$

Regression curve of x on y:*

$$\mu_1'(x)=a'\left(\frac{y}{b'}-1\right)+\frac{as'}{R}\cdot\frac{S_{R+1,\,s'+1}}{S_{R,\,s'}},$$

where

$$u=\frac{1}{a}+\frac{1}{a'}+Xy,$$

$$S_{R,\,s'}=u^{R-1}-\frac{aa'ls'}{R}.u^R+\ldots,$$

$$S_{R+1,\,s'+1}=u^R-aa'l.\frac{s'+1}{R+1}.u^{R+1}+\ldots.$$

* It can be easily shown that also the scedastic, clitic and kurtic curves are in the form of infinite series.

(c) *Application.* The results of the theory are illustrated on the distribution of barometric heights at Laudale and Southampton. The cell mid-ordinates and frequencies are computed, and the Goodness of Fit Test is applied to the whole surface as well as to the marginal totals*.

12. *Narumi's*† *System of Frequency Surfaces* (1923). Narumi starts his investigation on bivariate frequency surfaces from a consideration of the regression and scedastic curves. The regression curve need not be restricted to the curve of means; it can be any series of points defined in the same manner for each array. Let $x=f_1(y)$ and $y=f_2(x)$ be the two regression curves; and let $\frac{1}{F_1(y)}$ and $\frac{1}{F_2(x)}$ be the scales of measurement which will reduce the system to complete homoscedasticity. Then the most general functional equation to the frequency surface will be

$$z=\phi_1(y)\,\psi_1[\{x-f_1(y)\}\,F_1(y)]=\phi_2(x)\,\psi_2[\{y-f_2(x)\}\,F_2(x)].$$

The corresponding surfaces are determined for definite forms of $f_1(y)$, $f_2(x)$, $F_1(y)$ and $F_2(x)$.

The array distributions reduced to the regression curve as origin and reduced in scale owing to the heteroscedasticity, are similar and similarly situated curves. According to Narumi there is physically much to uphold this conception of the similarity of parallel arrays.

The most interesting cases considered are:

(i) Homoscedasticity and linear regression both ways → normal surface;

(ii) Scedasticity and regression linear both ways → Filon-Isserlis surface;

(iii) Scedastic and regression curves equilateral hyperbolae both ways

$$\rightarrow z=z_0(x+f_1)^{\gamma_1}(y+g_2)^{\gamma_2}.\,e^{\gamma(x+f_2)(y+g_1)};$$

the arrays are Pearson Type III curves;

(iv) Parabolic variance and linear regression → Pearson non-skew surface (see next section).

13. *Pearson's*‡ *Non-Skew Frequency Surfaces* (1923). An investigation by

* I would like to draw attention to the fact that by reducing the number of frequency groups in a bivariate distribution to about 25 broad groups and then applying the P, χ^2 Goodness of Fit Test, we are likely to get almost any value for P. Where a single surface has been fitted to an observed distribution too much significance should not be attached to the corresponding value of P in judging the descriptive power of that surface. We really want more comparative results: different equations fitted to the same observed distribution, the grouping not being altered throughout the investigation. The P's will then enable us to arrange the equations in order of merit as to their successful representation of the data. A comparison of this nature has been made by Pearson between "The Fifteen Constant Surface" and Rhodes' surface (see p. 128), and also between the Filon-Isserlis surface and the symmetrical surface described on p. 137 (see p. 124).

† "On the General Forms of Bivariate Frequency Distributions which are Mathematically Possible when Regression and Variation are subjected to Limiting Conditions," *Biometrika*, Vol. xv. 1923, pp. 77—88, 209—221.

‡ "Non-Skew Frequency Surfaces," *Biometrika*, Vol. xv. 1923, pp. 231—244.

Pearson on the range of frequency surfaces which would have symmetrical marginal distributions, led to the surface

$$z = \frac{N}{2\pi \cdot \sigma_1 \cdot \sigma_2 \cdot \sqrt{1-r^2}} \cdot \frac{n-1}{n-2} \cdot \frac{1}{\left[1 + \frac{1}{2(n-2)} \cdot \frac{1}{1-r^2}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1 \cdot \sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right]^n} \quad \ldots\ldots(55),$$

where

$$\frac{3(\beta_{20}-2)}{\beta_{20}-3} = n = \frac{3(\beta_{02}-2)}{\beta_{02}-3},$$

i.e. $\beta_{20} = \beta_{02}$.

The x-marginal distribution is

$$\phi_1(x) = \frac{N}{\sqrt{2\pi} \cdot \sigma_1} \cdot \frac{1}{\sqrt{n-2}} \cdot \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n-1)} \cdot \frac{1}{\left[1 + \frac{1}{2(n-2)} \cdot \frac{x^2}{\sigma_1^2}\right]^{n-\frac{1}{2}}}.$$

The surface has double linear regression and double parabolic variance. It has been shown later by Narumi that no other surface than Pearson's has these forms of regression and scedasticity combined (see previous section).

Regression curve of x on y:

$$\mu_1'(x) = -r \cdot \frac{\sigma_1}{\sigma_2} \cdot y.$$

Scedastic curve of x on y:

$$\sigma^2(x) = \sigma_1^2(1-r^2)\left\{1 + \frac{1}{2n+3}\left(1 - \frac{y^2}{\sigma_2^2}\right)\right\}.$$

When $\beta_2 = 3$, (55) reduces to the normal surface.

A number of special cases are considered when $\beta_2 < 3$:

(i) $n = 1$, $\beta_2 = 2{\cdot}25$: (55) → upper portion of a paraboloid;

(ii) $n = \frac{1}{2}$, $\beta_2 = 2{\cdot}1429$: (55) → upper half of an ellipsoid;

(iii) $n = 0$, $\beta_2 = 2{\cdot}000$: (55) → elliptic cylinder;

(iv) $0 > n \geqslant -\frac{1}{2}$, $2 > \beta_2 \geqslant 1{\cdot}8$: (55) → surface is cup-shaped; marginal totals are rectangles when $\beta_2 = 1{\cdot}8$;

(v) $-\frac{1}{2} > n \geqslant -1$, $1{\cdot}8 > \beta_2 \geqslant 1{\cdot}5$: (55) → also the marginal totals are now U-shaped.

14. *The Dissection of Frequency Surfaces.* In *Medd. från Lunds Astr. Obs.*, Ser. 2, No. 9, Charlier has dealt with the dissection of a bivariate distribution into two normal components with zero correlation:

$$\phi_1(x, y) = \frac{Z_0}{2\pi \cdot \sigma_1^2} \cdot e^{-\frac{1}{2}\left[\frac{(x-m_1)^2 + (y-n_1)^2}{\sigma_1^2}\right]},$$

and

$$\phi_2(x, y) = \frac{Z_0'}{2\pi \cdot \sigma_2^2} \cdot e^{-\frac{1}{2}\left[\frac{(x-m_2)^2 + (y-n_2)^2}{\sigma_2^2}\right]}.$$

The dissection into normal components with elliptical contours having their

principal axes parallel to the coordinate axes, has been treated by Åkesson*; the quite general case of normal components with any direction of principal axes has been discussed by Charlier and Wicksell†. Here

$$\phi_1(x,y) = \frac{Z_0}{2\pi \,.\, \sigma_1 \,.\, \sigma_1' \sqrt{1-r^2}} \,.\, e^{-\frac{1}{2(1-r^2)} \cdot \left[\left(\frac{x-m_1}{\sigma_1}\right)^2 - 2r\left(\frac{x-m_1}{\sigma_1}\right)\left(\frac{y-n_1}{\sigma_1'}\right) + \left(\frac{y-n_1}{\sigma_1'}\right)^2\right]},$$

$$\phi_2(x,y) = \frac{Z_0'}{2\pi \,.\, \sigma_2 \,.\, \sigma_2' \sqrt{1-r'^2}} \,.\, e^{-\frac{1}{2(1-r'^2)} \cdot \left[\left(\frac{x-m_2}{\sigma_2}\right)^2 - 2r'\left(\frac{x-m_2}{\sigma_2}\right)\left(\frac{y-n_2}{\sigma_2'}\right) + \left(\frac{y-n_2}{\sigma_2'}\right)^2\right]}.$$

In this last paper general equations are given for the moments of a bivariate distribution in terms of the moments of any two components. The case of normal components is then worked out fully. For the moment coefficients up to the fourth order fifteen equations are obtained involving the twelve unknowns. There are thus certain identical relations between the moments; and these may be regarded as criteria for dissecting the distribution into two normal components.

The determination of the unknowns is shown to depend on the solution of equations of an order not higher than the third. Some special cases, for which the general analysis is not valid, are treated separately.

15. '*Mutually Consistent Multiple Regression Surfaces.*' *Camp*‡ (1925). The object of Professor Camp's study is to determine what forms of the regression surfaces and of the total regressions are mathematically consistent with one another; arbitrary forms for the regression surfaces may not be combined with arbitrary forms for the total regressions.

He confines his study to trivariate distributions and assumes the regression surfaces to be polynomials of the second or higher order. These simple assumptions generally lead to total regressions of the form:

$$y = \frac{\text{polynomial in } x}{\text{polynomial in } x}.$$

E.g. Let the regression of z on x, y be:

$$g(x,y) = a + by + cx + dxy,$$

i.e. all the partial regressions linear, then the regression of z on x is of the form:

$$a(x) = \frac{\text{parabola in } x}{\text{parabola in } x}.$$

If $$g(x,y) = a + by + cx + dxy + ex^2 + fyx^2,$$

then $$a(x) = \frac{\text{cubic in } x}{\text{cubic in } x}.$$

The first of these expressions for $g(x, y)$ is subjected to a detailed treatment, as it is of the form assumed by Isserlis in his paper on the partial correlation ratio.

* "On the Dissection of Correlation Surfaces," *Arkiv för Mat., Astr. och Fysik*, Bd. 11, No. 16, 1916, pp. 1—16.

† "On the Dissection of Frequency Functions," *Arkiv för Mat., Astr. och Fysik*, Bd. 18, No. 6, 1923, pp. 1—64.

‡ *Biometrika*, Vol. XVII. 1925, pp. 443—458.

16. "*On Treating Skew Correlation.*" *Van Uven** (1925—29). The method followed by Van Uven in analysing skew correlation is equivalent to the principle of translation underlying the frequency curves of Edgeworth and of Kapteyn, viz. if x be the directly observed quantity, to find that function $f(x)$ of x which will be normally distributed. Whereas Edgeworth and Kapteyn assumed definite forms for $f(x)$, Van Uven† has developed a scheme for determining $f(x)$ graphically. This method is now extended to the treatment of correlation.

If the observed variates, x and y, are not normally correlated, the problem is to construct two functions t and t' of x and y which will follow the normal law and which will give as high a measure as possible of the correlation between x and y. Each of the new variables may involve both x and y, but it is generally possible, with the use of certain transformations, to express t (or t') as a function only of x (or only of y).

Let $x_{p-h/2}$ and $y_{q-k/2}$ be the mid-points of the pth x-array of y's and of the qth y-array of x's respectively; h and k the grouping units of x and y; and suppose p to vary from 1 to n, q from 1 to n'. Then in our usual notation:

$$n_{x_p} = \sum_{q=1}^{n'} n_{x_p y_q}, \quad n_{y_q} = \sum_{p=1}^{n} n_{x_p y_q}.$$

The relative frequencies of the x- and y-marginal totals will be $\frac{n_{x_p}}{N}$ and $\frac{n_{y_q}}{N}$ respectively; those of the x_p- and y_q-arrays will be $\frac{n_{x_p y}}{n_{x_p}}$ and $\frac{n_{xy_q}}{n_{y_q}}$ respectively.

Assume x and y, measured in terms of their standard deviations, to be normally distributed, i.e.

$$d\phi = \frac{\sqrt{1-r^2}}{2\pi} \cdot e^{-\frac{1}{2}(x^2 - 2rxy + y^2)} \cdot dx \cdot dy.$$

Write $\sqrt{1-r^2} \cdot x = z$ and $y - rx = \zeta$, then

$$d\phi = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}z^2} \cdot dz \times \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\zeta^2} \cdot d\zeta = du \cdot dv,$$

where $u = \theta(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{1}{2}w^2} \cdot dw$, and $v = \theta(\zeta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta} e^{-\frac{1}{2}w^2} \cdot dw$.

Similarly, by writing $\sqrt{1-r^2} \cdot y = z'$ and $x - ry = \zeta'$, we get

$$d\phi = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}z'^2} \cdot dz' \times \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\zeta'^2} \cdot d\zeta' = du' \cdot dv',$$

where

$$u' = \theta(z') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z'} e^{-\frac{1}{2}w^2} \cdot dw, \text{ and } v' = \theta(\zeta') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta'} e^{-\frac{1}{2}w^2} \cdot dw.$$

* *Proc. Kon. Ak. v. Wet.* (Amsterdam), Vol. XXVIII. Nos. 8—9, 1925, pp. 797—811; No. 10, pp. 919—935; Vol. XXIX. No. 4, 1926, pp. 580—590; Vol. XXXII. No. 4, 1929, pp. 408—413; see also "Skew Correlation between three and more Variables," *Proc. Kon. Ak. v. Wet.* Vol. XXXII. No. 6, 1929, pp. 793—807; No. 7, pp. 995—1007; No. 8, pp. 1085—1103.

† Kapteyn, J. C. and Van Uven, M. J.: *Skew Frequency Curves in Biology and Statistics.* 2nd Paper. Groningen, 1916, pp. 30—53.

It is easily seen that:

$$\theta(z_p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z(x_p)} e^{-\frac{1}{2}w^2} . dw \qquad = \frac{\sum_{i=1}^{p} n_{x_i}}{N},$$

$$\theta(z_q') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z'(y_q)} e^{-\frac{1}{2}w^2} . dw \qquad = \frac{\sum_{i=1}^{q} n_{y_i}}{N},$$

$$\theta(\zeta_{p,q-k/2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta(x_p, y_{q-k/2})} e^{-\frac{1}{2}w^2} . dw = \frac{\sum_{i=1}^{p} n_{x_i y_q}}{n_{y_q}} \text{ (approximately),}$$

and $$\theta(\zeta'_{p-h/2,q}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\zeta'(x_{p-h/2}, y_q)} e^{-\frac{1}{2}w^2} . dw = \frac{\sum_{i=1}^{q} n_{x_p y_i}}{n_{x_p}} \quad (\quad ,, \quad).$$

The conditions for normal correlation are: that the sets of values of the functions $z(x)$, $z'(y)$, $\zeta(x, y)$ and $\zeta'(x, y)$ obtained from these equations, have to satisfy linear relations:

$$z = ax + b, \quad \zeta = a'x + b'y + c', \text{ etc.}$$

If x and y follow a law of skew variation, two new variables t and t' are introduced as functions of z, z', ζ and ζ'—and thus as functions of x and y—to reduce the correlation to normality.

The method is illustrated on the correlation of height and volume of djati trees—a distribution that tends to be J-shaped in the one direction and thus representative of extremely skew correlation. The total number of observations is 916. The correlation is found to be ·948 as against ·831 calculated by the product-moment method. I find the values of the correlation ratios to be in fair agreement with the value of the relationship found by Van Uven; they are* $\eta_{yx} = ·916$ and $\eta_{xy} = ·938$, where x is the height and y the volume of the trees. No general conclusions can be drawn from this one example, but, on grounds of the agreement obtained, I think that the relatively small amount of labour demanded by the correlation ratio method in comparison with that involved in the application of Van Uven's method, is sufficient to establish, in the absence of further illustrations, the superiority of that method.

II.

D. *Further Analysis of Some of the Proposed Constructions.*

1. *The Array Moments of a Certain Double Hypergeometrical Series.* Let us consider the case of composite sampling specified on p. 123. It was stated there that the chance of s marked characters being drawn in the first sample and s' in the second, is

$$z(s, s') = \frac{n!\, n'!}{s!\, s'!\, (n-s)!\, (n'-s')!} \cdot \frac{(N-n-n')!}{N!} \cdot \frac{m!}{(m-s-s')!} \cdot \frac{(N-m)!}{(N-n-n'-m+s+s')!} \quad \ldots\ldots\ldots(56).$$

* No corrections for grouping or abruptness have been applied.

We have further, that the actual distribution of the terms of an array of second samples corresponding to a definite number s of marked characters in the first sample, is given by*

$$\frac{N-n-(m-s)}{N-n}\cdot\frac{N-n-1-(m-s)}{N-n-1}\cdots\frac{N-n-n'+1-(m-s)}{N-n-n'+1}$$
$$\times\left[1+n'\cdot\frac{m-s}{N-n-n'+1-(m-s)}\right.$$
$$+\frac{n'(n'-1)}{2!}\cdot\frac{(m-s)(m-s-1)}{(N-n-n'+1-(m-s))(N-n-n'+2-(m-s))}+\cdots$$
$$\left.+\frac{n'!}{s'!\,(n'-s')!}\cdot\frac{(m-s)(m-s-1)(m-s-2)\ldots(m-s-s'+1)}{(N-n-n'+1-(m-s))\ldots(N-n-n'+s'-(m-s))}+\ldots\right]$$
.....(57).

Write $\alpha=-n'$, $\beta=-(m-s)$, $\gamma=N-n-n'+1-(m-s)$; (57) is then seen to be the hypergeometrical series $F(\alpha,\beta,\gamma,1)$.

From the usual formulae for the moments of a hypergeometrical series, the moments of (57) can be readily written down and these will furnish us with the required expressions for the array moments.

Pearson finds the regression and scedasticity to be

$$\nu_1'(s')=n'\cdot\frac{m-s}{N-n}$$

and $$\nu_2(s')=n'\cdot\frac{N-n-n'}{N-n-1}\cdot\left[\frac{1}{4}-\left(\frac{m-s}{N-n}-\frac{1}{2}\right)^2\right]$$ respectively.

The third array moment is given by

$$\nu_3(s')=n'\cdot\frac{N-n-n'}{N-n-1}\cdot\frac{N-n-2n'}{N-n-2}\cdot\frac{m-s}{N-n}\cdot\left(1-\frac{m-s}{N-n}\right)\cdot\left(1-\frac{2(m-s)}{N-n}\right)$$
$$=\frac{N-n-2n'}{N-n-2}\cdot\left(1-\frac{2(m-s)}{N-n}\right)\cdot\nu_2(s').$$

Hence

$$\beta_1(s')\equiv\frac{\nu_3^2(s')}{\nu_2^3(s')}=\left(\frac{N-n-2n'}{N-n-2}\right)^2\cdot\frac{N-n-1}{n'(N-n-n')}\cdot\left[\frac{(N-n)^2}{(m-s)(N-n-(m-s))}-4\right]$$
.....(58).

The fourth array moment is

$$\nu_4(s')=n'\cdot\frac{N-n-n'}{N-n-1}\cdot\frac{m-s}{N-n}\cdot\left(1-\frac{m-s}{N-n}\right)\cdot\left[1-\frac{6(n'-1)(N-n-n'-1)}{(N-n-2)(N-n-3)}\right.$$
$$\left.+3(n'-2)\cdot\frac{m-s}{N-n}\cdot\left(1-\frac{m-s}{N-n}\right)\left\{1-\frac{n'-1}{N-n-2}\cdot\left(\frac{n'-10}{n'-2}+\frac{9}{N-n-3}\right)\right\}\right]$$
$$=\nu_2(s')\left[1-\frac{6(n'-1)(N-n-n'-1)}{(N-n-2)(N-n-3)}+3(n'-2)\frac{N-n-1}{n'(N-n-n')}\cdot\nu_2(s')\right.$$
$$\left.\times\left\{1-\frac{n'-1}{N-n-2}\cdot\left(\frac{n'-10}{n'-2}+\frac{9}{N-n-3}\right)\right\}\right].$$

* Pearson, Karl: "On a Certain Double Hypergeometrical Series and its Representation by Continuous Frequency Surfaces," *Biometrika*, Vol. XVI. 1924, p. 175.

Thus

$$\beta_2(s') \equiv \frac{\nu_4(s')}{\nu_2^2(s')} = \frac{N-n-1}{n'(N-n-2)(N-n-3)(N-n-n')} \times \left[3\{n'(N-n-n')(N-n+6) - 2(N-n)^2\} + \frac{(N-n)^2\{(N-n)(N-n-6n'+1)+6n'^2\}}{(m-s)(N-n-(m-s))} \right] \quad \ldots\ldots\ldots(59).$$

The relation between $\beta_1(s')$ and $\beta_2(s')$ can be obtained from equations (58) and (59). After some reductions I find it to be

$$\beta_1(s') + \frac{(N-n-3)(N-n-2n')^2}{(N-n-2)\{(N-n)(6n'-N+n-1)-6n'^2\}}\beta_2(s') = \frac{(N-n-1)(N-n-2n')^2\{3n'^2-(N-n)(3n'-2)\}}{n'(N-n-2)(N-n-n')\{(N-n)(N-n-6n'+1)+6n'^2\}} \quad \ldots(60).$$

Accordingly, β_1 and β_2 of the arrays lie on a straight line in the β-diagram; the curves of both $\beta_1(s')$ and $\beta_2(s')$ are U-shaped. Of course only a portion of these curves may correspond to appropriate values of s.

As an example we shall consider the case of whist correlation. Putting $N=52$, $n=n'=m=13$, we get:

$$\nu_2(s') = \tfrac{1}{171}(13-s)(26+s),$$

$$\beta_1(s') = \frac{(13+2s)^2}{(111)^2 \cdot \nu_2(s')} \quad \ldots\ldots\ldots(58)^{bis},$$

$$\beta_2(s') = 3{\cdot}081{,}081 - \frac{{\cdot}351{,}351}{\nu_2(s')} \quad \ldots\ldots\ldots(59)^{bis},$$

$$\beta_1(s') + {\cdot}351{,}351\,\beta_2(s') - 1{\cdot}027{,}027 = 0 \quad \ldots\ldots\ldots(60)^{bis}.$$

The curves $(58)^{bis}$, $(59)^{bis}$, $(60)^{bis}$ are shown in the accompanying diagram (p. 143). As s increases from 0 to 12, $\beta_1(s')$ increases from ·007 to ·500 and $\beta_2(s')$ decreases from 2·903 to 1·500; the rate of increase or decrease being small for low values of s. The diagram besides being instructive as to the form of the β-curves, shows why the Filon-Isserlis surface and the Pearson non-skew surface were found inadequate to represent the double hypergeometrical series*. Both these surfaces have similar parallel sections.

2. *The Third and Fourth Partial Moments of the Type AaAa Surface.* The equation of the surface can be written in the form

$$z = Z - \tfrac{1}{6}[q_{30}Z_{30} + 3q_{21}Z_{21} + 3q_{12}Z_{12} + q_{03}Z_{03}] + B_{20}Z_{40} + Q_{31}Z_{31} + 3Q_{22}Z_{22} + Q_{13}Z_{13} + B_{02}Z_{04} \quad \ldots\ldots\ldots(61),$$

where

$$Z = \frac{1}{2\pi\sqrt{1-r^2}} \cdot e^{-\frac{1}{2(1-r^2)}(x^2-2rxy+y^2)},$$

$$Z_{m,n} = \frac{d^{m+n}Z}{dx^m \cdot dy^n}, \qquad B_{20} = \tfrac{1}{24}(\beta_{20}-3),$$

$$Q_{31} = \tfrac{1}{6}(q_{31}-3r), \quad Q_{22} = \tfrac{1}{12}(q_{22}-1-2r^2), \text{ etc.}$$

* See p. 124. See remarks, *Biometrika*, Vol. XVI. p. 185.

The regression and scedastic curves of (61) have been dealt with in Section C, 6. The third and fourth partial moments will now be considered, firstly about the mean of the surface and then about the regression curve as origin. These moments have, in fact, been found before *, but not about the line of means. Moreover, the expressions can be put into forms much simpler than those given by Wicksell. It will thus be possible to perform the numerical applications more readily.

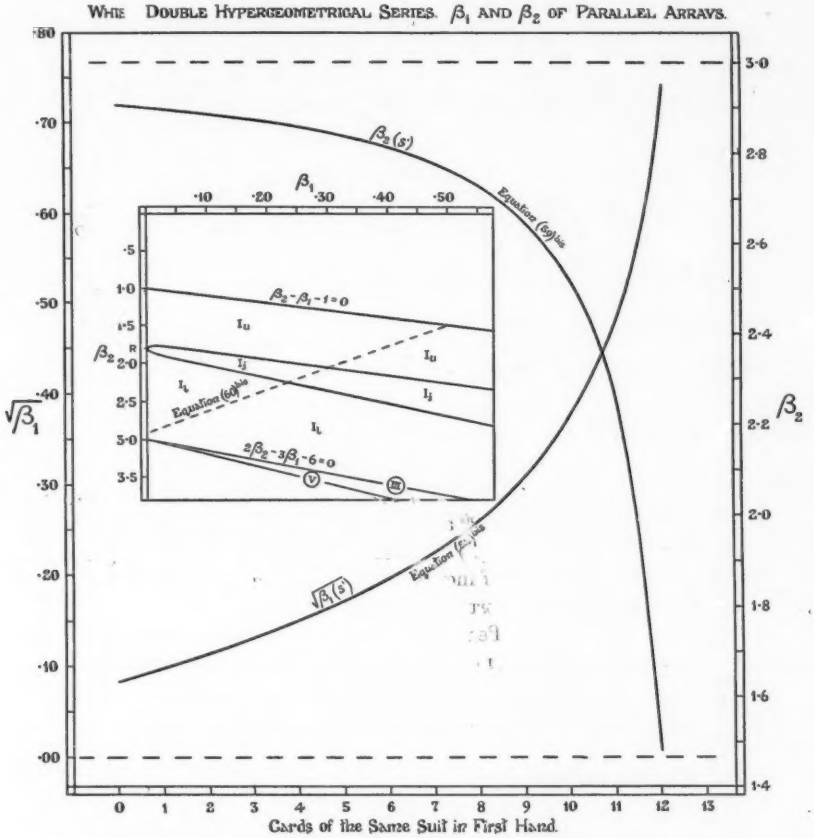

Diagram (3).

The following results are restated as we shall have to refer to them later on.

y-marginal Curve:

$$z_y = \tau_1 + \sqrt{\tfrac{2}{3}} \cdot \sqrt{\beta_{01}} \cdot \tau_4 + \sqrt{\tfrac{5}{24}} (\beta_{02} - 3) \tau_5$$
$$\equiv \tau_1 + a_4 \tau_4 + a_5 \tau_5 \ldots\ldots\ldots\

Regression Curve of x on y:

$$\mu_1'(x) = ry + \frac{\sqrt{\frac{3}{2}}(q_{12} - r\sqrt{\beta_{01}})\tau_3 + \sqrt{\frac{3}{8}}(q_{13} - r\beta_{02})\tau_4}{z_y}$$

$$\equiv ry + \frac{b_3\tau_3 + b_4\tau_4}{z_y} \quad \ldots\ldots(63)$$

$$\equiv ry + \frac{A_y}{z_y}.$$

Scedastic Curve of x on y:

$$\mu_2(x) = (1-r^2) - \frac{\sqrt{2}[r(q_{12} - r\sqrt{\beta_{01}}) - (q_{21} - rq_{12})]\tau_2 + \sqrt{6}[r(q_{13} - r\beta_{02}) - \frac{1}{2}(q_{22} - 1 - r^2(\beta_{02} - 1))]\tau_3}{z_y} - \left(\frac{A_y}{z_y}\right)^2$$

$$\equiv (1-r^2) - \frac{c_2\tau_2 + c_3\tau_3}{z_y} - \left(\frac{A_y}{z_y}\right)^2 \quad \ldots\ldots(64)$$

$$\equiv (1-r^2) - \left(\frac{B_y}{z_y}\right) - \left(\frac{A_y}{z_y}\right)^2.$$

(i) *The third partial Moment.*

Write $\int_{-\infty}^{+\infty} Z\,dx = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}y^2} \equiv v$ (say), $\frac{d^s}{dy^s}(v) \equiv v_s$.

Consider the integral $\int_{-\infty}^{+\infty} z \cdot x^3 dx = z_y \cdot \mu_3'(x)$.

We have $\int_{-\infty}^{+\infty} Z \cdot x^3 dx = v[r^3(y^3 - 3y) + 3ry]$,

$$\int_{-\infty}^{+\infty} Z_{30} \cdot x^3 dx = -6v, \quad \int_{-\infty}^{+\infty} Z_{21} \cdot x^3 dx = -6vr(y^2 - 1),$$

$$\int_{-\infty}^{+\infty} Z_{12} \cdot x^3 dx = -3v[r^2(y^4 - 6y^2 + 3) + (y^2 - 1)],$$

$$\int_{-\infty}^{+\infty} Z_{03} \cdot x^3 dx = v_3[r^3(y^3 - 3y) + 3ry] + 9rv_2[r^2(y^2 - 1) + 1] + 6r^3v[1 - 3y^2],$$

$$\int_{-\infty}^{+\infty} Z_{40} \cdot x^3 dx = 0, \quad \int_{-\infty}^{+\infty} Z_{31} \cdot x^3 dx = 6vy,$$

$$\int_{-\infty}^{+\infty} Z_{22} \cdot x^3 dx = 6rv(y^3 - 3y),$$

$$\int_{-\infty}^{+\infty} Z_{13} \cdot x^3 dx = 3v[(y^3 - 3y)(1 - r^2 + r^2y^2) - 6r^2y(y^2 - 1) + 6yr^2],$$

$$\int_{-\infty}^{+\infty} Z_{04} \cdot x^3 dx = v_4[r^3(y^3 - 3y) + 3ry] + 12v_3r[r^2(y^2 - 1) + 1] + 36v_2r^3y - 24vr^3y.$$

Accordingly:

$$\mu_3'(x) = 3(1 - r^2 + r^2y^2)A_y - 3ryB_y + [r^3(y^3 - 3y) + 3ry]z_y + v[6y(Q_{31} - 6rQ_{22} + 3r^2Q_{13} - 4r^3B_{02}) + \sqrt{\beta_{10}} - r^3\sqrt{\beta_{01}} - 3r(q_{21} - rq_{12})].$$

Using the expressions for $\mu_1'(x)$ and $\mu_2'(x)$, we get

$$\mu_3(x) = \frac{[\sqrt{\beta_{10}} - r^3\sqrt{\beta_{01}} - 3r(q_{21} - rq_{12})]\tau_1 + 6\sqrt{2}[Q_{31} - 6rQ_{22} + 3r^2Q_{13} - 4r^3B_{02})\tau_2}{z_y}$$
$$+ 3\left(\frac{A_y}{z_y}\right)\left(\frac{B_y}{z_y}\right) + 2\left(\frac{A_y}{z_y}\right)^3$$
$$\equiv \frac{d_1\tau_1 + d_2\tau_2}{z_y} + 3\left(\frac{A_y}{z_y}\right)\left(\frac{B_y}{z_y}\right) + 2\left(\frac{A_y}{z_y}\right)^3 \quad \ldots\ldots(65)$$
$$\equiv \left(\frac{C_y}{z_y}\right) + 3\left(\frac{A_y}{z_y}\right)\left(\frac{B_y}{z_y}\right) + 2\left(\frac{A_y}{z_y}\right)^3.$$

(ii) *The fourth partial Moment.*

Consider the integral $\int_{-\infty}^{+\infty} z \,.\, x^4 dx = z_y \,.\, \mu_4'(x)$.

We have $\int_{-\infty}^{+\infty} Z \,.\, x^4 dx = v[3(1-r^2)^2 + 6r^2(1-r^2)y^2 + r^4y^4]$,

$$\int_{-\infty}^{+\infty} Z_{30} \,.\, x^4 dx = -24vry, \quad \int_{-\infty}^{+\infty} Z_{21} \,.\, x^4 dx = -12v[y + r^2(y^3 - 3y)],$$
$$\int_{-\infty}^{+\infty} Z_{12} \,.\, x^4 dx = -4[v_2\{r^3(y^3 - 3y) + 3ry\} + 6v_1r\{r^2(y^2-1)+1\} + 6vr^3y],$$
$$\int_{-\infty}^{+\infty} Z_{03} \,.\, x^4 dx = v_3[3(1-r^2)^2 + 6r^2(1-r^2)y^2 + r^4y^4] + 12v_2r[r^3(y^3 - 3y) + 3ry]$$
$$+ 36v_1r^2[r^2(y^2-1)+1] + 24vr^4y,$$
$$\int_{-\infty}^{+\infty} Z_{40} \,.\, x^4 dx = 24v, \quad \int_{-\infty}^{+\infty} Z_{31} \,.\, x^4 dx = 24vr(y^2-1),$$
$$\int_{-\infty}^{+\infty} Z_{22} \,.\, x^4 dx = 12v[r^2(y^4 - 6y^2 + 3) + (y^2-1)],$$
$$\int_{-\infty}^{+\infty} Z_{13} \,.\, x^4 dx = -4[v_3\{r^3(y^3 - 3y) + 3ry\} + 9v_2r\{r^2(y^2-1)+1\} - 18vr^3y^2 + 6vr^3],$$
$$\int_{-\infty}^{+\infty} Z_{04} \,.\, x^4 dx = v_4[3(1-r^2)^2 + 6r^2(1-r^2)y^2 + r^4y^4] + 16v_3r[r^3(y^3 - 3y) + 3ry]$$
$$+ 72v_2r^2[r^2(y^2-1)+1] - 96vr^4y^2 + 24vr^4.$$

Combining these expressions, we get

$$z_y \,.\, \mu_4'(x) = [3(1-r^2)^2 + 6r^2(1-r^2)y^2 + r^4y^4]z_y + 4[r^3(y^3 - 3y) + 3ry]A_y$$
$$- 6[r^2(y^2-1)+1]B_y + 4ry \,.\, C_y + [(\beta_{20} - 3)(1 - 4r^2) - 3r^4(\beta_{02} - 3)$$
$$+ 6r^2(q_{22} - 1 - 2r^2) - 4r(q_{31} - r\beta_{20}) - 4r^3(q_{13} - r\beta_{02})].$$

Hence, if the regression curve be the origin:

$$\mu_4(x) = 3(1-r^2)^2 + \frac{\{[(\beta_{20} - 3)(1 - 4r^2) - 3r^4(\beta_{02} - 3) + 6r^2(q_{22} - 1 - 2r^2)\} - 4r(q_{31} - r\beta_{20}) - 4r^3(q_{13} - r\beta_{02})]\tau_1}{z_y}$$
$$- 6(1-r^2)\left[\frac{B_y}{z_y} + \left(\frac{A_y}{z_y}\right)^2\right] - 4\left(\frac{A_y}{z_y}\right)\left(\frac{C_y}{z_y}\right) - 6\left(\frac{A_y}{z_y}\right)^2\left(\frac{B_y}{z_y}\right) - 3\left(\frac{A_y}{z_y}\right)^4$$
$$\equiv 3(1-r^2)^2 + \frac{e_1\tau_1}{z_y} - 6(1-r^2)\left[\frac{B_y}{z_y} + \left(\frac{A_y}{z_y}\right)^2\right] - 4\left(\frac{A_y}{z_y}\right)\left(\frac{C_y}{z_y}\right)$$
$$- 6\left(\frac{A_y}{z_y}\right)^2\left(\frac{B_y}{z_y}\right) - 3\left(\frac{A_y}{z_y}\right)^4 \quad \ldots\ldots(66).$$

This can be written as

$$\mu_4(x) = 6(1-r^2)\cdot\mu_2(x) - 3(1-r^2)^2 + \frac{D_y}{z_y} - 4\left(\frac{A_y}{z_y}\right)\left(\frac{C_y}{z_y}\right) - 6\left(\frac{A_y}{z_y}\right)^2\left(\frac{B_y}{z_y}\right) - 3\left(\frac{A_y}{z_y}\right)^4,$$

where $D_y = e_1\tau_1$.

From equations (64), (65) and (66) the clitic and kurtic curves can be obtained.

3. *The Logarithmically Transformed Normal Curve**. The object of this section is to analyse more fully the relations between the fourth and lower moment coefficients of the curve:

$$y = y_0 \cdot \frac{1}{x} \cdot e^{-\frac{1}{2}\left(\frac{\log_{10} x - l}{s}\right)^2} \qquad \text{.....(67).}$$

We have from p. 120:

$$\mu_1' = e^{bl + \frac{1}{2}b^2s^2},$$

$$\mu_2 = e^{2bl + b^2s^2}\cdot[e^{b^2s^2} - 1],$$

$$\mu_3 = e^{3bl + \frac{9}{2}b^2s^2}\cdot[e^{3b^2s^2} - 3e^{b^2s^2} + 2],$$

$$\mu_4 = e^{4bl + 2b^2s^2}\cdot[e^{6b^2s^2} - 4e^{3b^2s^2} + 6e^{b^2s^2} - 3].$$

The mode of (67) is given by

$$x_{\text{mode}} = e^{bl - b^2s^2}.$$

Write

$$e^{b^2s^2} = \lambda,$$

then:

$$\left.\begin{aligned} \text{Skewness} = \chi &= \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}} = \frac{1-\lambda^{-\frac{3}{2}}}{\sqrt{\lambda - 1}} \\ \beta_1 &= \lambda^2(\lambda+3) - 4 \\ \eta = \beta_2 - 3 &= \lambda^2(\lambda^2 + 2\lambda + 3) - 6 \end{aligned}\right\} \text{.....(68).}$$

Differentiating χ, we find it has a maximum value for $\lambda = 1{\cdot}7201$ which gives $\chi_{\max} = {\cdot}6561$. The skewness of the curve therefore ranges from 0 to ·6561.

* After I had completed the analysis of this section a paper by G. K. Davies on "The Analysis of Frequency Distributions" appeared in the *Journ. Am. Stat. Ass.* December, 1929. The author refers therein to a study by himself on "The Logarithmic Curve of Distribution" in the same *Journal*, December, 1925—a study of which I had been unaware.

In the first of these papers the parameters corresponding to our l and s, are found from the mean and standard deviation computed by replacing the class marks by their logarithms. In the second paper the quartile dispersion is adopted as the basis of the method of fitting. Two illustrations of the application of this method are given; the total number of observations being 10 and 32. I have not had occasion to test the efficiency of this method in relation to the method of moments (described in Section C, 2), but it is conceivable that for such small numbers the method of quartiles might be adequate.

On p. 359 of his second paper Davies writes:

"Since [the logarithmic normal] can be varied to give any required degree of skewness...."

The author gives two equations corresponding to our (68) as well as a small β-diagram indicating the relation of the log. normal curve to the Pearson curves, yet he makes the above remark. The relation between β_1 and β_2 expressed by equations (68) will always exist, no matter by what method the parameters of the curve have been determined, and thereby the skewness of the curve is strictly defined. In particular, it cannot describe mesokurtic distributions.

DIAGRAM SHOWING THE RELATION BETWEEN β_1 AND β_2 FOR THE LOGARITHMICALLY TRANSFORMED NORMAL CURVE.

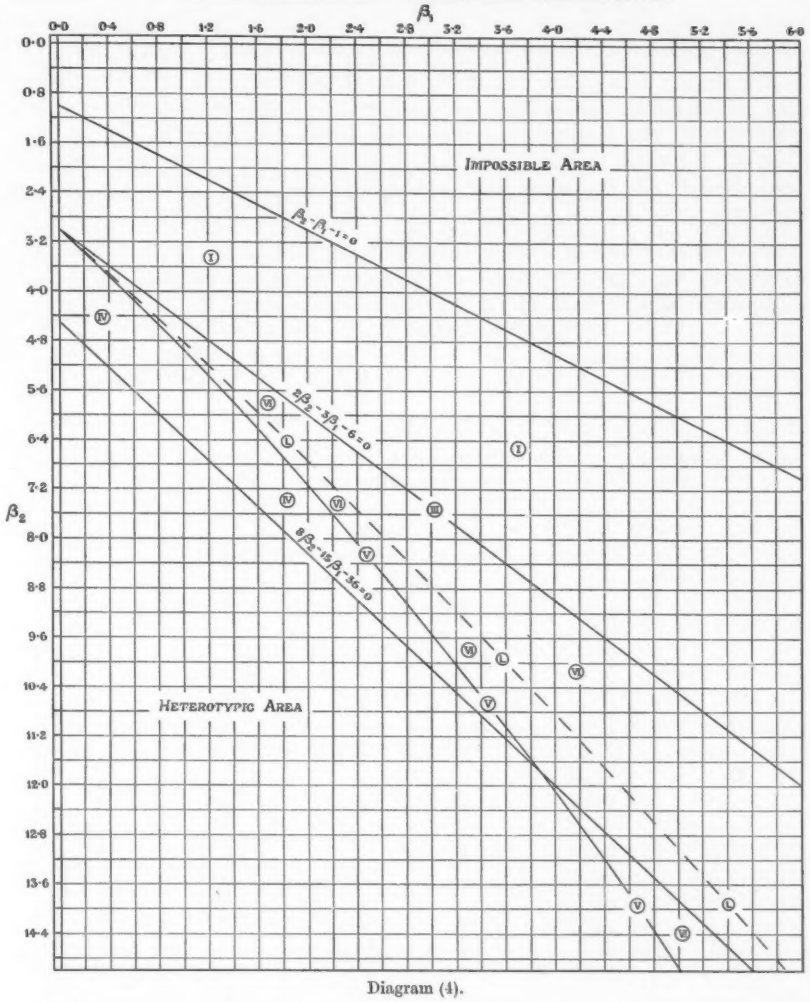

Diagram (4).

The relations between the first four moments of (67) are expressed by β_1 and β_2, both being functions of λ only. Eliminating λ between equations (68) we get *

$$\beta_1^4 + 12\beta_1^3 + 156\beta_1^2 + 64\beta_1 - \eta^3 + 12\eta^2 - 36\eta + 9\beta_1^2 . \eta - 6\beta_1 . \eta^2 - 108\beta_1 . \eta = 0 \quad \ldots\ldots(69).$$

* The maximum value of χ and the equation corresponding to our (69) are incorrect as given on p. 195, *Biometrika*, Vol. IV.

This curve, plotted in relation to the Pearson Types, is shown as the broken line *LL* in the accompanying β_1, β_2 diagram; it passes about midway through the Type VI area. The diagram will be of use in ascertaining from the β's of an observed distribution whether the data follow a law of the form (67), or not.

We proceed to a comparison of the log. normal curve with the corresponding Type VI by fitting both curves to a distribution whose β's satisfy approximately relations (68). The outcome of such a comparison is of practical importance in so far as the log. normal curve is easier to apply than the Type VI; the particular advantage of the former curve being that the cell frequencies are directly obtainable.

The second column of Table (1) shows the distribution of 1951 readings of the height of the barometer, at Greenwich, on the first day for a reading of 30·1″—30·2″ on the third day (see Table III, p. 154). The constants are

$$\text{Mean height of barometer} = 30{\cdot}0049'',$$

$$\sigma = 2{\cdot}419{,}847 \text{ (unit} = \tfrac{1}{10}''), \quad \beta_1 = {\cdot}712{,}997, \quad \beta_2 = 4{\cdot}307{,}638.$$

TABLE (1).

Distribution of Barometric Heights, represented by a Log. Normal and a Type VI Curve.

Barometric Height	Observed Frequency	Theor. Freq. Log. Normal	Theor. Freq. Type VI
30·75	1		
30·65	1	3·4	3·4
30·55	10		
30·45	32	34·0	34·2
30·35	111	127·2	127·1
30·25	214	252·6	252·6
30·15	386	336·8	336·3
30·05	365	342·1	342·0
29·95	288	288·3	288·6
29·85	199	212·8	213·2
29·75	129	143·0	143·1
29·65	86	89·6	89·6
29·55	62	53·4	53·3
29·45	26	30·6	30·5
29·35	17	17·1	17·0
29·25	10	9·3	9·3
29·15	9	5·1	5·1
29·05	2	2·7	2·7
28·95	2	1·4	1·4
28·85	1	1·6	1·6
Totals	1951	1951	1951
χ^2	—	23·334	23·484
P	—	·038	·037

The β-diagram shows that the condition (69) is approximately fulfilled. The observed value of β_1 in equations (68) gives $\beta_2 = 4{\cdot}2908$.

The constants of the log. normal curve are, in tenths of inches as units:

$$\xi_1 = \text{mean-start} = 8{\cdot}813{,}403,$$
$$l = {\cdot}929{,}362, \quad s = {\cdot}117{,}082.$$

The constants of the Type VI curve

$$y = y_0 (x-a)^{q_2} x^{-q_1}$$

are

$$q_1 = 50{\cdot}369{,}290, \qquad q_2 = 15{\cdot}683{,}288,$$
$$a = 15{\cdot}513{,}656, \quad \log y_0 = 57{\cdot}414{,}365,$$
$$\text{Mean-start} = 7{\cdot}918{,}337.$$

In columns three and four of Table (1), the theoretical frequencies are exhibited; a very close agreement between the two theories is manifest.

The question now arises, how accurately will the one curve reproduce the other for β's satisfying exactly the relation (69)? Suppose we take

$$\beta_1 = {\cdot}961{,}000, \quad \sigma = 2{\cdot}400{,}000,$$
$$\beta_2 = 4{\cdot}756{,}100, \quad N = 1000.$$

The log. normal curve has for its constants:

$$\xi_1 = 7{\cdot}589{,}466, \quad l = {\cdot}859{,}515, \quad s = {\cdot}134{,}077.$$

The Type VI, with origin at the mean:

$$y = y_0 \left(1 + \frac{x}{a_2}\right)^{q_2} \left(1 + \frac{x}{a_1}\right)^{-q_1}$$

has

$$a_1 = 22{\cdot}407{,}476, \quad a_2 = 6{\cdot}594{,}502,$$
$$q_1 = 38{\cdot}769{,}330, \quad q_2 = 10{\cdot}115{,}484, \quad \log y_0 = 2{\cdot}225{,}343.$$

Corresponding ordinates of the two curves at unit intervals of the argument are given in Table (2). The argument is measured from the start of the log. normal curve and the correspondence is about the means of the curves*.

The close agreement obtained between the two theories in this example, as in the previous one, indicates that for all practical purposes the Type VI curve may be replaced by the log. normal when β_1 and β_2 satisfy relation (69). The examples suggest too that for fairly high values of the β's, a small deviation from (69) hardly affects the form of this curve. However, it remains to be investigated generally within what range of deviation from (69) the two curves will still give equally reliable results.

The high contact the log. normal curve has at its start—a theoretical disadvantage—is brought out clearly in these illustrations by the distance from start to mean.

* [The Log. Normal Curve can only be looked upon as a possibly easier means of determining subrange frequencies in such a case. Its form is deduced from the Weber-Fechner Law in psychology, which can have no application to meteorological phenomena. The agreement is really only established with a Type VI curve, which lies on an infinitesimal portion of the area to which this curve applies. ED.]

TABLE (2).

Corresponding Ordinates of the Log. Normal and Type VI Curves (Special Case).

Log. Normal	Type VI
·11	·06
7·38	7·14
51·12	51·53
126·19	126·54
179·15	178·89
183·54	183·17
153·22	153·10
111·87	111·95
74·63	74·76
46·81	46·90
28·14	28·17
16·42	16·42
9·40	9·38
5·30	5·29
2·97	2·96
1·66	1·65
·92	·92
·51	·51
·29	·28

E. *An Examination of the Adequacy of the Mathematical Surfaces. Graphical Analysis and Specification of Observed Data.*

1. *Data.* When I began investigating the present problem, Professor Pearson kindly placed at my disposal a number of correlation tables showing the distribution of contemporaneous barometric heights at various meteorological stations. The total number of observations, in the tables, varied from about 1800 to about 3000. I tested Narumi's surfaces on one of these distributions by fitting the theoretical regression and scedastic curves, but found the surfaces to be inadequate. Eventually, the 15-Constant Surface (Type AaAa) was resorted to. The process of fitting and of constructing the contours was arduous enough; in addition, however, the result did not repay the labour. The scantiness of the material made it impossible to judge the accuracy of the graduation. A similar insufficiency of observations is found in the examples on which the surfaces have been tested in earlier papers. As a first requisite, therefore, for obtaining results that would be of some value, distributions had to be found in which the irregularities of sampling would be less pronounced.

Table I shows the number of marriages contracted in Australia, 1907—14, arranged according to the ages of bride and bridegroom in 3-year groups. It was formed from Table LIV of Knibbs' work: *The Mathematical Theory of Population, of its Character and Fluctuations, and of the Factors which influence them*, Melbourne, 1917, pp. 190—191, where the ages are given by single years. Unspecified cases, brides over 85 and bridegrooms over 90 were rejected. In these data, as in

practically all marriage statistics, there is unquestionably a misstatement of ages by persons under 21 years of age, the chief motive of such a misstatement being to avoid legal requirements. No attempt was made to adjust the numbers.

In Table II the number of single births (male and female) in Australia, 1922—26, is tabulated according to the ages of father and mother in 3-year and 2-year groups respectively. The table was compiled from the corresponding tables in the *Australian Demography Bulletins*, Nos. 40, 41, 42, 43, and 44, the unspecified cases being again omitted.

The distribution of barometric heights on alternate days at Greenwich, 1848—1926, for the whole year, summer months (March 21—September 21) and winter months, is shown in Tables III, IV, and V respectively. These tables were drawn up from the barometric readings published in *Astronomical and Meteorological and Magnetical Observations made at the Royal Observatory, Greenwich*, for each of the 79 years. The marginal totals of Table IV, and also those of Table V, are not identical because the last reading in the summer or winter period was not correlated with the first reading in the period for the next year.

Finally, Table VI exhibits the distribution of a set of measurements made by W. Johannsen on the length and breadth of beans. The table is reproduced from Wicksell's study, *The Correlation Function of Type A and the Regression of its Characteristics*, p. 40.

The choice of one or two of the distributions might be regarded by some readers as ill advised. There were, however, no alternatives; other data, with equally large numbers, that would be better suited for illustrations, could not be found. Looked at from the skewness of the distributions, which varies from slightly abnormal to considerably abnormal, the data are representative of statistics of common occurrence.

In each of the tables only the central values of the groups are recorded.

2. *Regression, Scedastic, Clitic and Kurtic Curves.* The problem of skew correlation has repeatedly been approached from a consideration of the form of the regression curves. But we are concerned not so much with these discussions as with testing the regression, scedastic, clitic, and kurtic curves associated with the theoretical surfaces. In particular, an examination of the clisy and kurtosis of the arrays will provide us with a practical test as to the generality of Narumi's hypothesis, namely, that the array distributions reduced to a common origin and scale are similar and similarly situated curves.

The statistical measures computed from the distributions and to be used for specifying them, are given in Tables I (a), I (b), I (c) to VI (c). The higher moments of the extreme arrays, where the observations are relatively few, were not calculated. The standard deviations of the arrays, as well as those of the marginal totals, are expressed in terms of the grouping units. Sheppard's corrections have been applied to all the momental constants. Often they reduced the values of the correlation ratios below that of the corresponding correlation coefficient. Corrections for

TABLE I.

Number of Marriages arranged according to the Ages of the Contracting Parties, Australia, 1907—14.

Age of Bridegroom. (Central Values of 3-year groups.) ⟵ $+y$	Age of Bride. (Central Values of 3-year groups.) ⟶ $+x$																									
	12·5	15·5	18·5	21·5	24·5	27·5	30·5	33·5	36·5	39·5	42·5	45·5	48·5	51·5	54·5	57·5	60·5	63·5	66·5	69·5	72·5	75·5	78·5	81·5	84·5	Totals
16·5	—	69	167	47	7	4	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	294
19·5	2	651	5577	3619	836	201	67	26	10	4	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	10995
22·5	2	1066	15064	27394	12677	3313	920	331	124	67	21	13	6	2	—	1	—	—	—	—	—	—	—	—	—	61001
25·5	—	615	9024	24028	24110	10664	3018	951	380	157	55	26	16	6	3	1	—	—	—	—	—	—	—	—	—	73054
28·5	—	284	4619	13619	16407	12865	5502	1930	765	321	98	52	24	8	5	—	—	1	1	—	—	—	—	—	—	56501
31·5	—	125	1941	6141	7942	7540	5515	2551	1055	407	136	73	34	12	2	3	—	1	—	—	—	—	—	—	—	33478
34·5	—	66	922	2952	4196	4130	3389	2538	1349	590	241	125	42	24	2	1	1	—	—	1	—	—	—	—	—	20569
37·5	—	41	506	1536	2347	2658	2282	1918	1494	914	308	146	88	21	12	8	—	1	—	1	—	—	—	—	—	14281
40·5	—	29	214	716	1135	1410	1489	1248	1138	991	506	256	110	50	16	8	4	—	—	—	—	—	—	—	—	9320
43·5	—	10	100	352	585	753	829	912	828	749	554	326	144	55	23	9	2	4	—	1	—	—	—	—	—	6236
46·5	—	9	73	217	337	443	548	556	691	561	511	438	203	95	47	22	8	7	4	—	—	—	—	—	—	4770
49·5	1	4	49	101	228	264	310	342	454	540	354	350	313	172	55	41	21	13	8	—	—	—	—	—	—	3620
52·5	—	3	17	56	76	111	168	170	230	286	262	240	233	174	81	45	19	7	8	3	1	—	—	—	—	2190
55·5	—	—	10	30	49	71	99	129	155	182	170	200	192	144	96	76	20	15	13	2	—	1	1	—	—	1655
58·5	—	1	2	17	28	44	45	53	81	95	94	134	137	103	85	91	43	25	14	4	2	2	—	—	—	1100
61·5	—	1	4	6	19	27	28	31	53	82	55	77	88	99	65	60	43	29	24	13	4	1	—	1	—	810
64·5	—	1	—	4	14	20	18	34	31	43	42	69	64	60	58	40	44	43	34	20	7	2	1	—	—	649
67·5	—	—	1	7	6	8	13	13	21	24	36	35	50	45	42	44	32	39	39	23	3	2	3	1	—	487
70·5	—	—	—	—	4	8	13	9	15	22	16	20	22	36	24	26	23	29	22	25	6	4	2	—	—	326
73·5	—	—	—	1	5	5	1	5	5	17	10	12	18	16	15	12	14	16	21	24	10	1	3	—	—	211
76·5	—	—	—	2	—	1	4	3	2	5	3	3	9	9	8	16	7	7	13	6	12	7	1	—	1	119
79·5	—	—	1	2	2	—	2	2	2	2	3	6	4	3	3	8	3	3	3	6	10	3	2	3	—	73
82·5	—	—	—	—	—	—	—	—	—	3	1	2	4	2	3	—	5	2	1	1	—	—	3	—	—	27
85·5	—	—	—	—	—	—	1	—	—	—	1	1	1	3	—	1	2	—	—	—	1	2	—	1	—	14
88·5	—	—	—	—	—	1	—	—	—	—	—	—	3	—	—	—	—	—	1	—	—	—	—	—	—	5
Totals	5	2975	38291	80847	71010	44541	24261	13752	8883	6062	3478	2605	1805	1139	645	513	291	242	206	130	56	25	16	6	1	301785

TABLE II.

Number of Single Births arranged according to the Ages of the Parents, Australia, 1922—26.

→ $+x$ Age of Mother. (Central Values of 2-year groups.)

↓ $+y$ Age of Father. (Central values of 3-year groups.)	13·0	15·0	17·0	19·0	21·0	23·0	25·0	27·0	29·0	31·0	33·0	35·0	37·0	39·0	41·0	43·0	45·0	47·0	49·0	51·0	53·0	55·0	Totals
16·5	1	15	93	52	16	2	2	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	181
19·5	—	58	1268	3308	2208	719	232	82	36	16	3	4	2	—	—	—	—	—	—	—	—	—	7936
22·5	—	55	1666	7999	12949	10808	4506	1569	691	275	162	50	32	15	5	7	—	—	—	—	—	—	40789
25·5	1	30	787	5224	12977	20682	20814	11711	4452	1860	769	374	168	79	27	7	2	—	—	—	—	—	79964
28·5	—	20	411	2528	7321	14518	21087	22889	17712	7519	2998	1293	587	298	100	37	7	2	1	—	—	—	99328
31·5	1	4	154	1142	3881	8061	12974	17886	20951	19603	10770	4124	1675	677	267	108	20	4	1	—	—	—	102303
34·5	—	3	89	538	1760	4189	7265	9967	13381	15719	17551	12336	5013	1984	606	207	58	2	2	—	—	—	90670
37·5	—	—	49	275	826	1886	3397	5792	7699	9546	11917	12730	11449	5713	1655	532	113	26	4	—	—	—	73609
40·5	—	1	27	127	394	886	1608	2498	3983	5518	6385	8088	8788	8600	4423	1326	236	35	6	—	1	—	52930
43·5	—	—	15	57	195	422	761	1161	1863	2510	4193	4601	5514	5775	4651	2958	734	86	11	—	—	—	35507
46·5	—	2	5	32	112	206	367	644	860	1318	1710	2637	3286	3460	3180	2516	1224	239	16	2	1	—	21817
49·5	—	2	4	21	62	114	197	311	551	615	905	1281	1679	2284	1920	1637	844	282	68	4	—	—	12781
52·5	—	1	—	7	21	60	102	157	209	348	526	615	849	1016	1115	997	483	177	32	2	—	—	6717
55·5	—	—	2	8	18	32	54	78	108	168	246	358	500	525	519	490	354	104	18	1	2	2	3587
58·5	—	—	1	2	9	18	26	42	69	76	131	156	217	345	260	233	155	59	21	1	—	—	1821
61·5	—	—	—	—	6	11	14	26	36	55	70	102	80	143	134	114	75	32	10	3	—	—	911
64·5	—	—	—	1	2	2	12	10	24	22	38	44	56	75	70	72	40	15	6	—	—	—	489
67·5	—	—	—	1	—	3	2	6	4	7	18	16	20	39	24	25	12	4	2	—	—	—	183
70·5	—	—	1	—	1	1	1	2	5	4	6	16	9	11	10	9	5	3	1	—	—	—	85
73·5	—	—	1	—	—	—	—	3	2	2	5	6	2	5	3	5	3	1	—	—	—	—	38
76·5	—	—	—	—	—	—	—	—	2	—	2	3	4	6	5	3	—	—	—	—	—	—	25
79·5	—	—	—	—	—	—	2	—	2	—	1	—	2	—	1	—	—	1	—	—	—	—	9
82·5	—	—	—	—	—	—	—	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	2
Totals	3	191	4573	21322	42758	62620	73423	74834	72640	65182	58407	48834	39932	31050	18975	11283	4365	1072	199	13	4	2	631682

TABLE III.

Barometric Heights at Greenwich on Alternate Days. Whole Year, 1848—1926.

$\longrightarrow +x$ First day. Height in Inches (Central Values).

Third Day. Height in Inches (Central Values). $+y \longleftarrow$

	30·75	30·65	30·55	30·45	30·35	30·25	30·15	30·05	29·95	29·85	29·75	29·65	29·55	29·45	29·35	29·25	29·15	29·05	28·95	28·85	28·75	28·65	28·55	28·45	28·35	Totals
30·75	2	1	2	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
30·65	—	1	3	1	3	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	13
30·55	2	1	6	14	17	17	3	8	2	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	73
30·45	1	4	10	51	53	48	31	23	15	13	1	5	2	1	—	—	—	—	—	—	—	—	—	—	—	258
30·35	1	5	21	48	91	113	84	70	52	31	29	7	2	6	3	—	—	—	—	—	—	—	—	—	—	563
30·25	—	—	16	53	107	177	214	191	142	82	75	39	23	17	7	2	1	1	—	—	1	—	—	—	—	1148
30·15	1	1	10	32	111	214	386	365	288	199	129	86	62	26	17	10	9	2	2	1	—	—	—	—	—	1951
30·05	—	—	1	25	75	217	363	554	543	436	284	193	108	63	40	21	11	6	8	2	—	1	—	—	—	2951
29·95	—	—	1	20	51	152	312	589	737	670	489	297	205	119	50	29	14	8	3	1	3	—	—	—	—	3750
29·85	—	—	2	5	29	93	230	473	654	707	608	457	262	170	121	56	29	15	3	5	1	1	—	—	—	3921
29·75	—	—	1	6	14	59	132	270	545	613	659	535	345	251	125	73	39	15	11	3	3	—	—	—	—	3699
29·65	—	—	—	—	5	26	100	196	336	493	468	522	384	253	172	97	62	31	21	5	3	2	—	—	—	3176
29·55	—	—	—	1	2	11	41	102	192	272	382	374	303	244	155	103	69	38	20	6	13	3	2	—	—	2333
29·45	—	—	—	1	3	9	21	53	123	168	254	275	221	193	170	97	89	33	23	10	5	3	1	—	—	1752
29·35	—	—	—	1	2	6	19	28	49	109	152	160	165	164	129	100	63	39	24	7	6	7	1	2	—	1233
29·25	—	—	—	—	—	—	8	14	29	63	83	91	120	106	106	69	51	28	20	9	7	7	1	1	—	813
29·15	—	—	—	—	—	2	1	7	25	40	45	57	56	68	54	77	38	26	20	14	5	4	2	—	—	541
29·05	—	—	—	—	—	1	1	6	4	10	24	37	39	33	35	36	17	9	15	6	4	5	—	—	—	282
28·95	—	—	—	—	—	—	1	—	11	8	13	21	15	23	15	21	18	12	12	5	5	8	—	1	—	189
28·85	—	—	—	—	—	—	—	1	1	2	—	8	9	8	14	11	13	5	2	2	4	—	2	—	—	82
28·75	—	—	—	—	—	—	—	—	—	3	3	7	7	4	11	5	9	6	—	2	—	2	1	—	—	60
28·65	—	—	—	—	—	—	—	1	—	—	1	4	3	2	7	4	8	5	2	3	—	—	2	—	1	43
28·55	—	—	—	—	—	—	—	—	1	1	—	—	1	—	2	1	1	2	3	—	—	—	—	—	—	12
28·45	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	1	—	—	—	—	—	—	—	4
28·35	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
Totals	7	13	73	258	563	1148	1951	2951	3749	3921	3700	3176	2333	1752	1233	813	542	282	189	81	60	43	12	4	1	28855

TABLE IV.

Barometric Heights at Greenwich on Alternate Days. Summer Months, 1848—1926.

→ $+x$ — First Day. Height in Inches (Central Values).

Third Day. Height in Inches (Central Values). ↓ $+y$	30·45	30·35	30·25	30·15	30·05	29·95	29·85	29·75	29·65	29·55	29·45	29·35	29·25	29·15	29·05	28·95	28·85	28·75	28·65	Totals
30·45	2	2	3	2	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	10
30·35	—	11	20	18	13	7	5	5	—	—	—	—	—	—	—	—	—	—	—	79
30·25	3	19	54	68	67	48	28	26	11	8	2	2	—	—	—	—	—	—	—	336
30·15	1	20	79	176	194	148	103	57	33	20	6	4	1	—	—	—	—	—	—	842
30·05	2	13	79	185	340	344	256	175	99	39	26	10	7	1	—	3	1	—	—	1580
29·95	2	10	60	154	370	492	468	323	170	117	62	17	5	2	3	—	1	—	—	2256
29·85	—	3	23	128	279	456	479	416	305	163	88	55	22	2	4	—	—	—	—	2423
29·75	—	3	15	52	139	380	420	460	355	217	137	59	25	9	2	—	—	—	—	2273
29·65	—	—	4	31	109	190	325	314	319	223	146	76	40	22	5	2	1	—	—	1807
29·55	—	—	1	8	34	110	168	235	225	191	137	70	36	20	7	7	—	2	—	1251
29·45	—	—	—	6	15	53	91	142	157	123	95	74	34	25	5	4	2	1	—	827
29·35	—	—	—	2	11	18	44	77	72	65	72	54	35	16	8	4	—	—	—	478
29·25	—	—	—	—	3	6	23	25	34	43	35	39	22	12	5	2	1	—	1	251
29·15	—	—	—	—	1	4	12	14	15	15	10	12	18	7	4	3	1	2	—	118
29·05	—	—	—	—	—	—	2	6	10	9	4	7	4	2	1	—	—	—	1	46
28·95	—	—	—	—	—	—	2	4	4	1	1	3	3	3	3	—	1	—	—	25
28·85	—	—	—	—	—	—	1	—	2	1	1	1	—	—	—	1	—	—	—	7
28·75	—	—	—	—	—	—	—	—	1	1	—	1	—	1	—	—	1	—	—	5
28·65	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	1
Totals	10	81	338	830	1576	2256	2427	2279	1812	1236	822	484	252	122	47	27	9	5	2	14615

TABLE V.

Barometric Heights at Greenwich on Alternate Days. Winter Months, 1848—1926.

$\longrightarrow +x$ First Day. Height in Inches (Central Values).

$+y \longleftarrow$ Third Day. Height in Inches (Central Values).

	30·75	30·65	30·55	30·45	30·35	30·25	30·15	30·05	29·95	29·85	29·75	29·65	29·55	29·45	29·35	29·25	29·15	29·05	28·95	28·85	28·75	28·65	28·55	28·45	28·35	Totals
30·75	2	1	2	—	—	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	7
30·65	—	1	3	1	3	2	3	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	—	13
30·55	2	1	6	14	17	17	3	8	2	1	1	1	—	—	—	—	—	—	—	—	—	—	—	—	—	73
30·45	1	4	10	49	51	45	29	22	15	13	1	5	2	1	—	—	—	—	—	—	—	—	—	—	—	248
30·35	1	5	21	48	80	93	66	57	45	26	24	7	2	6	3	—	—	—	—	—	—	—	—	—	—	484
30·25	—	—	16	50	88	123	146	124	94	54	49	28	15	15	5	2	1	1	—	—	1	—	—	—	—	812
30·15	1	1	10	31	91	135	210	171	140	96	72	53	42	20	13	9	9	2	2	1	—	—	—	—	1	1109
30·05	—	—	1	23	62	138	178	214	199	180	109	94	69	37	30	14	10	6	5	1	—	1	—	—	—	1371
29·95	—	—	1	18	41	92	158	219	245	202	166	127	88	57	33	24	12	5	3	—	3	—	—	—	—	1494
29·85	—	—	2	5	26	70	102	194	198	228	192	152	99	82	66	34	27	11	3	5	1	1	—	—	—	1498
29·75	—	—	1	6	11	44	80	131	165	193	199	180	128	114	66	48	30	13	11	3	3	—	—	—	—	1426
29·65	—	—	—	—	5	22	69	87	146	168	154	203	161	107	96	57	40	26	19	4	3	2	—	—	—	1369
29·55	—	—	—	1	2	10	33	68	82	104	147	149	112	107	85	67	49	31	13	6	11	3	2	—	—	1082
29·45	—	—	—	1	3	9	15	38	70	77	112	118	98	98	96	63	64	28	19	8	4	3	1	—	—	925
29·35	—	—	—	1	2	6	17	17	31	65	75	88	100	92	75	65	47	31	20	7	6	7	1	2	—	755
29·25	—	—	—	—	—	—	8	11	23	40	58	57	77	71	67	47	39	23	18	8	7	6	1	1	—	562
29·15	—	—	—	—	—	2	1	6	21	28	31	42	41	58	42	59	31	22	17	13	3	4	2	—	—	423
29·05	—	—	—	—	—	1	1	6	4	8	18	27	30	29	28	32	15	8	15	6	4	4	—	—	—	236
28·95	—	—	—	—	—	—	1	—	11	6	9	17	14	22	12	18	15	9	12	4	5	8	—	1	—	164
28·85	—	—	—	—	—	—	—	1	1	1	—	6	8	7	13	11	13	5	1	2	4	—	2	—	—	75
28·75	—	—	—	—	—	—	—	—	—	3	3	6	6	4	10	5	8	6	—	1	—	2	1	—	—	55
28·65	—	—	—	—	—	—	—	1	—	—	1	4	3	2	7	4	8	5	1	3	—	—	2	—	1	42
28·55	—	—	—	—	—	—	—	—	1	1	—	—	1	—	2	1	1	2	3	—	—	—	—	—	—	12
28·45	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	1	1	1	—	—	—	—	—	—	—	4
28·35	—	—	—	—	—	—	—	—	—	—	—	—	—	1	—	—	—	—	—	—	—	—	—	—	—	1
Totals	7	13	73	248	482	810	1121	1375	1493	1494	1421	1364	1097	930	749	561	420	235	162	72	55	41	12	4	1	14240

TABLE VI.

Correlation of Length and Breadth of Beans.

→ $+x$ Length in mm. (Central Values). / ↓ $+y$ Breadth in mm. (Central Values).	17·0	16·5	16·0	15·5	15·0	14·5	14·0	13·5	13·0	12·5	12·0	11·5	11·0	10·5	10·0	9·5	Totals
9·125	—	2	—	—	3	—	—	—	—	—	—	—	—	—	—	—	5
8·875	4	8	17	19	—	—	—	—	—	—	—	—	—	—	—	—	48
8·625	2	23	101	156	93	23	2	—	—	—	—	—	—	—	—	—	400
8·375	—	18	105	494	574	227	56	9	—	—	—	—	—	—	—	—	1483
8·125	—	4	44	375	956	913	362	73	12	3	—	—	—	—	—	—	2742
7·875	—	—	7	81	385	871	794	330	89	19	3	—	—	—	—	—	2579
7·625	—	—	1	4	65	236	469	361	175	55	27	4	—	—	—	—	1397
7·375	—	—	—	—	6	23	91	137	124	78	37	22	11	—	1	—	530
7·125	—	—	—	—	—	1	13	18	28	35	25	32	11	6	1	—	170
6·875	—	—	—	—	—	—	—	1	9	8	21	12	13	7	1	—	72
6·625	—	—	—	—	—	—	—	—	—	—	2	—	1	4	3	—	10
6·375	—	—	—	—	—	—	—	—	—	1	—	—	—	1	1	1	4
Totals	6	55	275	1129	2082	2294	1787	929	437	199	115	70	36	18	7	1	9440

TABLE I (*a*).

Constants of the Distribution of Ages at Marriage of Bride and Bridegroom.

Age of Bride	Age of Bridegroom	—	—
—	—	—	$q_{21}=1\cdot576,067$
$\bar{x}=$ 25·721,621 yrs.	$\bar{y}=$ 29·383,065 yrs.	—	$q_{12}=1\cdot477,487$
$\sigma_1=$ 2·239,567*	$\sigma_2=$ 2·640,766*	$r=\cdot708,164$	$q_{31}=7\cdot069,828$
$\sqrt{\beta_{10}}=+2\cdot013,900$	$\sqrt{\beta_{01}}=+1\cdot963,079$	$\eta_{yx}=\cdot710,278$	$q_{22}=6\cdot345,730$
$\beta_{20}=$ 9·290,441	$\beta_{02}=$ 8·332,812	$\eta_{xy}=\cdot707,139$	$q_{13}=6\cdot297,500$

* Unit = three years.

TABLE I (*b*).

Constants of the Distribution of Age of Bride for a given Age of Bridegroom (Central Values).

Age of Bridegroom in years	Number of Observations	Mean Age of Bride in years	$\sigma(x)$ in three year units	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
16·5	294	18·540,818	—	—	—
19·5	10,995	20·066,711	·849,742	1·769,638	9·808,637
22·5	61,001	21·871,010	1·000,300	1·455,142	8·579,920
25·5	73,054	23·624,156	1·168,982	1·061,578	6·392,154
28·5	56,501	25·132,271	1·389,366	·887,181	5·152,802
31·5	33,478	26·637,494	1·614,018	·697,360	4·091,898
34·5	20,569	28·117,823	1·878,609	·624,843	3·550,400
37·5	14,281	29·720,222	2·107,620	·512,228	3·237,101
40·5	9,320	31·859,012	2·351,849	·340,680	2·787,481
43·5	6,236	33·782,393	2·502,022	·216,977	2·719,178
46·5	4,770	35·790,566	2·761,511	·138,784	2·717,721
49·5	3,620	38·067,125	2·994,725	·072,250	2·691,692
52·5	2,190	40·509,590	3·089,567	−·059,573	2·738,178
55·5	1,655	42·383,987	3·248,947	−·038,079	2·715,627
58·5	1,100	45·246,362	3·494,342	−·186,099	2·627,812
61·5	810	47·014,814	3·721,307	−·142,120	2·673,850
64·5	649	49·110,170	3·964,999	−·152,912	2·410,205
67·5	487	51·450,719	4·016,722	−·326,509	2·647,233
70·5	326	52·374,233	4·172,709	−·248,391	2·220,272
73·5	211	54·372,038	4·368,691	−·402,175	2·352,104
76·5	119	57·096,638	4·441,906	—	—
79·5	73	56·226,026	—	—	—
82·5	27	56·055,557	—	—	—
85·5	14	54·973,682	—	—	—
88·5	5		—	—	—

TABLE I (*c*).

Constants of the Distribution of Age of Bridegroom for a given Age of Bride (Central Values).

Age of Bride in years	Number of Observations	Mean Age of Bridegroom in years	$\sigma(y)$ in three year units	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
12·5	5	24·177,180	—	—	—
15·5	2,975		1·635,113	2·080,477	10·440,077
18·5	38,291	24·677,352	1·454,074	1·888,116	9·515,703
21·5	80,847	26·073,156	1·506,919	1·888,895	9·604,855
24·5	71,010	27·841,233	1·655,005	1·829,298	8·990,077
27·5	44,541	30·022,665	1·877,199	1·633,245	7·751,172
30·5	24,261	32·621,553	2·174,789	1·337,310	6·214,616
33·5	13,752	35·311,518	2·437,164	1·018,662	4·932,579
36·5	8,883	38·367,276	2·705,296	·710,997	3·932,594
39·5	6,062	41·474,925	2·942,219	·585,660	3·921,733
42·5	3,478	44·399,655	2·944,590	·405,820	3·821,951
45·5	2,605	47·263,533	3·114,894	·228,121	3·537,179
48·5	1,805	50·459,004	3·273,171	·138,528	3·698,322
51·5	1,139	53·537,751	3·216,616	− ·032,434	3·528,247
54·5	645	56·397,675	3·112,739	− ·221,998	3·515,073
57·5	513	58·576,023	3·056,301	− ·205,697	3·729,017
60·5	291	61·747,422	2·878,866	− ·016,835	3·213,237
63·5	242	63·024,792	2·856,280	− ·789,109	4·195,793
66·5	206	64·907,766	2·690,352	− ·620,610	4·391,585
69·5	130	67·730,769	2·456,385	− 1·491,384	7·436,024
72·5	56	71·892,858	2·280,592	—	—
75·5	25	71·940,000	—	—	—
78·5	16	73·760,868	—	—	—
81·5	6		—	—	—
84·5	1		—	—	—

TABLE II (*a*).

Constants of the Distribution of Ages of Parents at Birth of Child.

Age of Mother	Age of Father	—	—
—	—	—	q_{21} = ·275,142
$\bar{x}$ = 29·529,067 yrs.	$\bar{y}$ = 33·500,298 yrs.	—	q_{12} = ·302,215
σ_1 = 3·083,148*	σ_2 = 2·495,111†	r = ·734,944	q_{31} = 1·823,465
$\sqrt{\beta_{10}}$ = + ·317,180	$\sqrt{\beta_{01}}$ = + ·724,331	η_{yx} = ·732,763	q_{22} = 1·837,166
β_{20} = 2·430,327	β_{02} = 3·624,169	η_{xy} = ·747,517	q_{13} = 2·065,256

* Unit = two years. † Unit = three years.

TABLE II (*b*).

Constants of the Distribution of Age of Mother for a given Age of Father (Central Values).

Age of Father in years	Number of Observations	Mean Age of Mother in years	$\sigma(x)$ in two year units	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
16·5	181	17·895,028	·859,357	—	—
19·5	7,936	19·915,574	1·105,597	1·420,909	7·540,025
22·5	40,789	21·932,114	1·371,058	1·167,144	6·493,849
25·5	79,964	24·184,358	1·554,972	·692,084	4·622,581
28·5	99,328	26·362,838	1·758,487	·357,841	3·802,815
31·5	102,303	28·529,886	1·939,264	·041,134	3·182,722
34·5	90,670	30·677,270	2·134,633	−·226,993	2·863,438
37·5	73,609	32·737,206	2·296,582	−·399,857	2·831,788
40·5	52,930	34·594,068	2·424,931	−·552,244	2·922,367
43·5	35,507	36·091,222	2·514,318	−·601,512	3·074,234
46·5	21,817	37·183,894	2·605,658	−·666,868	3·222,510
49·5	12,781	37·756,592	2·668,569	−·710,871	3·367,968
52·5	6,717	38·087,986	2·647,144	−·708,986	3·260,171
55·5	3,587	38·203,234	2·728,736	−·719,828	3·480,831
58·5	1,821	38·205,930	2·733,062	−·703,866	3·393,736
61·5	911	37·886,938	2·866,767	−·532,496	2·904,552
64·5	489	38·059,304	2·796,094	−·623,135	3·011,293
67·5	183	37·885,246	2·703,871	—	—
70·5	85	36·952,942	—	—	—
73·5	38	36·368,422	—	—	—
76·5	25		—	—	—
79·5	9	36·444,444	—	—	—
82·5	2		—	—	—

TABLE II (*c*).

Constants of the Distribution of Age of Father for a given Age of Mother (Central Values).

Age of Mother in years	Number of Observations	Mean Age of Father in years	$\sigma(y)$ in three year units	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
13·0	3	23·396,907	—	—	—
15·0	191		—	—	—
17·0	4,573	23·568,009	1·510,032	2·295,506	13·785,722
19·0	21,322	24·700,263	1·450,460	1·786,585	8·683,177
21·0	42,758	26·289,678	1·526,565	1·646,047	7·835,439
23·0	62,620	27·851,709	1·559,772	1·519,692	7·189,398
25·0	73,423	29·436,369	1·591,399	1·474,353	7·112,446
27·0	74,834	31·100,715	1·637,989	1·371,596	6·605,027
29·0	72,640	32·769,054	1·685,861	1·328,398	6·592,213
31·0	65,182	34·491,348	1·710,695	1·205,339	6·013,309
33·0	58,407	36·360,651	1·773,938	1·165,181	6·101,650
35·0	48,834	38·241,246	1·823,740	1·042,839	5·597,504
37·0	39,932	40·184,538	1·848,826	·854,808	5·055,700
39·0	31,050	42·152,754	1·920,485	·788,422	4·917,428
41·0	18,975	44·147,904	1·912,000	·610,390	4·662,052
43·0	11,283	46·089,471	1·923,478	·451,864	4·638,673
45·0	4,365	48·097,938	1·901,278	·350,883	4·230,476
47·0	1,072	49·947,762	1·921,691	·365,142	4·707,391
49·0	199	51·509,175	—	—	—
51·0	13		—	—	—
53·0	4		—	—	—
55·0	2		—	—	—

TABLE III (*a*).

Constants of the Distribution of Barometric Heights (Whole Year) on Alternate Days.

First Day	Third Day	—	—
—	—	—	q_{21} = ·152,059
$\bar{x}$ = 29·780,934″	$\bar{y}$ = 29·780,931″	—	q_{12} = ·169,381
σ_1 = 3·079,720*	σ_2 = 3·080,000*	r = ·580,721	q_{31} = 1·881,406
$\sqrt{\beta_{10}}$ = + ·450,793	$\sqrt{\beta_{01}}$ = + ·451,289	η_{yx} = ·583,316	q_{22} = 1·817,042
β_{20} = 3·397,763	β_{02} = 3·398,787	η_{xy} = ·581,703	q_{13} = 1·865,340

* In $\frac{1}{10}$ inches.

TABLE III (*b*).

Constants of the Distribution of Barometric Heights (Whole Year) on First Day for a given Reading on Third Day (Central Values).

Reading on Third Day in inches	Number of Observations	Mean Height on First Day in inches	$\sigma(x)$ in $\frac{1}{10}$ inches	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
30·75	7	30·420,000	—	—	—
30·65	13		—	—	—
30·55	73	30·300,685	2·020,996	—	—
30·45	258	30·240,310	2·196,790	·868,163	3·770,511
30·35	563	30·160,302	2·349,221	·648,474	3·495,585
30·25	1148	30·065,679	2·431,305	·760,496	4·057,606
30·15	1951	30·004,946	2·419,847	·844,391	4·307,638
30·05	2951	29·929,261	2·395,152	·819,870	4·445,142
29·95	3750	29·876,960	2·275,430	·588,481	4·032,045
29·85	3921	29·811,719	2·376,758	·556,345	3·686,299
29·75	3699	29·753,866	2·368,077	·414,211	3·579,364
29·65	3176	29·694,490	2·530,774	·458,017	3·329,710
29·55	2333	29·633,069	2·661,476	·533,459	3·594,689
29·45	1752	29·586,301	2·741,230	·318,388	3·112,579
29·35	1233	29·534,023	2·897,096	·278,185	3·357,779
29·25	813	29·487,023	2·884,793	·363,886	2·782,546
29·15	541	29·442,606	3·076,942	·169,269	2·676,531
29·05	282	29·422,695	2·995,550	·275,610	3·027,310
28·95	189	29·361,111	3·483,078	—	—
28·85	82	29·304,878	2·899,690	—	—
28·75	60	29·338,333	2·941,605	—	—
28·65	43	29·226,744	—	—	—
28·55	12		—	—	—
28·45	4	29·285,294	—	—	—
28·35	1		—	—	—

TABLE III (*c*).

Constants of the Distribution of Barometric Heights (Whole Year) on Third Day for a given Reading on First Day (Central Values).

Reading on First Day in inches	Number of Observations	Mean Height on Third Day in inches	$\sigma(y)$ in $\frac{1}{10}$ inches	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
30·75	7	30·460,000	—	—	—
30·65	13		—	—	—
30·55	73	30·322,603	1·844,786	—	—
30·45	258	30·242,248	2·071,353	·988,036	4·441,460
30·35	563	30·174,334	2·086,388	·659,948	3·860,262
30·25	1148	30·085,279	2·216,594	·596,379	4·064,244
30·15	1951	30·004,075	2·229,517	·645,631	3·865,614
30·05	2951	29·937,326	2·225,411	·616,284	4·236,583
29·95	3749	29·865,631	2·298,247	·606,296	4·192,548
29·85	3921	29·804,068	2·387,431	·549,445	3·883,501
29·75	3700	29·747,784	2·479,312	·371,584	3·389,310
29·65	3176	29·690,460	2·588,024	·457,062	3·733,761
29·55	2333	29·644,299	2·715,802	·390,583	3·426,421
29·45	1752	29·598,858	2·769,057	·223,362	3·224,785
29·35	1233	29·538,970	2·941,199	·282,386	3·266,638
29·25	813	29·477,921	2·988,265	·121,441	2·842,953
29·15	542	29·444,834	3·118,031	·275,851	3·201,995
29·05	282	29·412,411	3·227,022	·284,802	3·142,534
28·95	189	29·389,153	3·165,014	—	—
28·85	81	29·330,247	3·348,281	—	—
28·75	60	29·375,000	3·109,796	—	—
28·65	43	29·240,698	—	—	—
28·55	12	28·852,941	—	—	—
28·45	4		—	—	—
28·35	1		—	—	—

TABLE IV (*a*).

Constants of the Distribution of Barometric Heights (Summer Months) on Alternate Days.

First Day	Third Day	—	—
—	—	—	q_{21} = ·151,949
$\bar{x}$ = 29·790,075″	$\bar{y}$ = 29·790,753″	—	q_{12} = ·169,116
σ_1 = 2·415,849″	σ_2 = 2·407,294″	r = ·534,545	q_{31} = 1·687,358
$\sqrt{\beta_{10}}$ = +·448,990	$\sqrt{\beta_{01}}$ = +·432,215	η_{yx} = ·534,897	q_{22} = 1·649,758
β_{20} = 3·274,413	β_{02} = 3·214,970	η_{xy} = ·533,048	q_{13} = 1·630,514

TABLE IV (*b*).

Constants of the Distribution of Barometric Heights (Summer Months) on First Day for a given Reading on Third Day (Central Values).

Reading on Third Day in inches	Number of Observations	Mean Height on First Day in inches	$\sigma(x)$ in inches	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
30·45	10	30·141,011	—	—	—
30·35	79		—	—	—
30·25	336	30·036,012	2·047,375	·689,178	3·228,771
30·15	842	29·998,812	1·879,447	·734,010	3·560,910
30·05	1580	29·928,608	1·973,564	·855,845	4·854,866
29·95	2256	29·875,310	1·923,916	·536,856	3·782,482
29·85	2423	29·816,364	2·003,191	·469·880	3·182,739
29·75	2273	29·761,835	1·959,977	·317,173	3·096,858
29·65	1807	29·709,989	2·131,423	·383,309	3·087,158
29·55	1251	29·654,396	2·148,063	·498,913	3·518,774
29·45	827	29·613,000	2·230,345	·402,282	3·191,061
29·35	478	29·567,573	2·313,959	·145,890	2·615,369
29·25	251	29·514,940	2·334,763	·235,434	3·101,169
29·15	118	29·484,746	—	—	—
29·05	46	29·497,826	—	—	—
28·95	25	29,422,000	—	—	—
28·85	7	29·380,769	—	—	—
28·75	5		—	—	—
28·65	1		—	—	—

TABLE IV (*c*).

Constants of the Distribution of Barometric Heights (Summer Months) on Third Day for a given Reading on First Day (Central Values).

Reading on First Day in inches	Number of Observations	Mean Height on Third Day in inches	$\sigma(y)$ in inches	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
30·45	10	30·145,604	—	—	—
30·35	81		—	—	—
30·25	338	30·075,740	1·584,353	·349,398	2·971,343
30·15	830	30·001,084	1·741,844	·528,892	3·349,466
30·05	1576	29·937,056	1·805,284	·603,407	3·690,422
29·95	2256	29·867,509	1·850,561	·430,994	3·331,418
29·85	2427	29·808,879	2·027,150	·544,597	3·651,131
29·75	2279	29·757,635	2·122,282	·362,806	3·326,920
29·65	1812	29·706,457	2·148,778	·405,566	3·597,764
29·55	1236	29·669,498	2·209,315	·246,041	3·256,883
29·45	822	29·634,550	2·158,241	·106,433	2·854,964
29·35	484	29·570,661	2·378,887	·078,428	2·988,715
29·25	252	29·525,000	2·407,206	-·043,209	2·529,909
29·15	122	29·469,672	2·171,182	—	—
29·05	47	29·460,638	—	—	—
28·95	27	29·442,593	—	—	—
28·85	9	29·362,500	—	—	—
28·75	5		—	—	—
28·65	2		—	—	—

TABLE V (*a*).

Constants of the Distribution of Barometric Heights (Winter Months) on alternate Days.

First Day	Third Day	—	—
—	—	—	q_{21}= ·103,602
$\bar{x}$=29·771,552″	$\bar{y}$=29·770,850″	—	q_{12}= ·119,394
σ_1= 3·634,793*	σ_2= 3·640,723*	r= ·600,961	q_{31}=1·609,191
$\sqrt{\beta_{10}}$= + ·377,097	$\sqrt{\beta_{01}}$= + ·378,687	η_{y_x}= ·606,313	q_{22}=1·553,029
β_{20}= 2·861,522	β_{02}= 2·862,281	η_{x_y}= ·604,689	q_{13}=1·599,874

* In $\frac{1}{10}$ inches.

TABLE V (*b*).

Constants of the Distribution of Barometric Heights (Winter Months) on First Day for a given Reading on Third Day (Central Values).

Reading on Third Day in inches	Number of Observations	Mean Height on First Day in inches	$\sigma(x)$ in $\frac{1}{10}$ inches	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
30·75	7	30·420,000	—	—	—
30·65	13		—	—	—
30·55	73	30·300,685	—	—	—
30·45	248	30·239,113	2·226,484	·855,755	3·690,966
30·35	484	30·166,116	2·438,117	·677,793	3·444,574
30·25	812	30·077,956	2·563,449	·823,536	4·153,907
30·15	1109	30·009,603	2·759,459	·852,819	3·938,848
30·05	1371	29·930,015	2·803,430	·760,497	3·728,625
29·95	1494	29·879,451	2·721,317	·590,782	3·534,137
29·85	1498	29·804,206	2·878,610	·523,148	3·233,538
29·75	1426	29·741,164	2·897,747	·358,568	3·047,418
29·65	1369	29·674,032	2·964,589	·372,890	2·910,239
29·55	1082	29·608,410	3·134,490	·393,193	3·005,976
29·45	925	29·562,432	3·108,770	·168,964	2·746,087
29·35	755	29·512,781	3·193,847	·209,155	3·166,914
29·25	562	29·474,555	3·090,935	·330,929	3·000,836
29·15	423	29·430,851	3·139,860	·115,633	2·700,373
29·05	236	29·408,051	3·093,051	·145,121	2·890,636
28·95	164	29·351,829	3·557,265	—	—
28·85	75	29·287,333	—	—	—
28·75	55	29·340,909	—	—	—
28·65	42	29·233,333	—	—	—
28·55	12	29·285,294	—	—	—
28·45	4		—	—	—
28·35	1		—	—	—

TABLE V (*c*).

Constants of the Distribution of Barometric Heights (Winter Months) on Third Day for a given Reading on First Day (Central Values).

Reading on First Day in inches	Number of Observations	Mean Height on Third Day in inches	$\sigma(y)$ in $\frac{1}{10}$ inches	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
30·75	7	30·460,000	—	—	—
30·65	13		—	—	—
30·55	73	30·322,603	—	—	—
30·45	248	30·244,758	2·080,683	1·029,967	4·545,211
30·35	482	30·179,876	2·155,189	·703,291	3·864,196
30·25	810	30·089,259	2·431,190	·628,479	3·788,161
30·15	1121	30·006,289	2·530,527	·654,585	3·513,264
30·05	1375	29·937,636	2·625,525	·582,473	3·749,428
29·95	1493	29·862,793	2·843,795	·606,873	3·567,920
29·85	1494	29·796,252	2·876,403	·473,828	3·337,469
29·75	1421	29·731,984	2·956,578	·277,531	2·894,125
29·65	1364	29·669,208	3·063,120	·350,034	3·174,959
29·55	1097	29·615,907	3·167,617	·291,203	2·926,329
29·45	930	29·567,312	3·180,523	·088,028	2·843,682
29·35	749	29·518,491	3·236,809	·241,729	3·005,220
29·25	561	29·456,774	3·192,760	·068,007	2·711,597
29·15	420	29·437,619	3·339,718	·222,960	2·937,186
29·05	235	29·402,766	3·319,181	·287,181	3·120,079
28·95	162	29·380,247	3·166,841	—	—
28·85	72	29·320,833	—	—	—
28·75	55	29·375,454	—	—	—
28·65	41	29·245,122	—	—	—
28·55	12		—	—	—
28·45	4	29·102,941	—	—	—
28·35	1		—	—	—

TABLE VI (*a*).

Constants of the Distribution of Length and Breadth of Beans.

Length of Beans	Breadth of Beans	—	—
—	—	—	$q_{21} = -·653,267$
$\bar{x} = 14·404,608$ mm.	$\bar{y} = 7·975,530$ mm.	—	$q_{12} = -·502,495$
$\sigma_1 = 1·799,562$*	$\sigma_2 = 1·359,481$†	$r = ·781,142$	$q_{31} = 3·641,829$
$\sqrt{\beta_{10}} = -·910,569$	$\sqrt{\beta_{01}} = -·440,832$	$\eta_{y_x} = ·771,995$	$q_{22} = 3·167,011$
$\beta_{20} = 4·862,944$	$\beta_{02} = 3·654,374$	$\eta_{x_y} = ·772,270$	$q_{13} = 3·072,626$

* In $\frac{1}{2}$ mm. units. † In $\frac{1}{2}$ mm. units.

TABLE VI (*b*).

Constants of the Distribution of Length of Beans for a given Breadth (Central Values).

Breadth of Beans in mm.	Number of Observations	Mean Length of Beans in mm.	$\sigma(x)$ in $\frac{1}{2}$ mm.	$\sqrt{\beta_1(x)}$	$\beta_2(x)$
9·125	5	16·047,170	—	—	—
8·875	48		·899,218	—	—
8·625	400	15·510,000	·975,329	− ·045,354	2·925,986
8·375	1483	15·132,165	·964,928	− ·238,931	3·479,772
8·125	2742	14·736,689	1·025,400	− ·261,031	3·456,580
7·875	2579	14·258,822	1·122,755	− ·246,864	3·390,668
7·625	1397	13·777,380	1·267,017	− ·447,534	3·505,273
7·375	530	13·135,849	1·599,946	− ·539,225	3·316,993
7·125	170	12·370,588	1·663,320	− ·052,792	2·593,873
6·875	72	11·763,889	1·554,314	—	—
6·625	10	11·357,143	—	—	—
6·375	4		—	—	—

TABLE VI (*c*).

Constants of the Distribution of Breadth of Beans for a given Length (Central Values).

Length of Beans in mm.	Number of Observations	Mean Breadth of Beans in mm.	$\sigma(y)$ in $\frac{1}{4}$ mm.	$\sqrt{\beta_1(y)}$	$\beta_2(y)$
17·0	6	8·584,017	—	—	—
16·5	55		·872,333	—	—
16·0	275	8·442,273	·867,956	− ·451,028	3·132,322
15·5	1129	8·296,391	·821,161	+ ·030,368	3·049,913
15·0	2082	8·153,698	·846,552	− ·105,878	3·645,700
14·5	2294	8·000,436	·834,188	− ·058,086	3·219,250
14·0	1787	7·845,621	·878,903	− ·116,996	3·274,500
13·5	929	7·712,998	·885,527	− ·065,917	3·157,425
13·0	437	7·571,224	·953,572	− ·321,257	3·187,621
12·5	199	7·434,045	1·035,346	− ·183,445	3·815,952
12·0	115	7·288,044	1·113,176	—	—
11·5	70	7·189,286	—	—	—
11·0	36	7·097,222	—	—	—
10·5	18	6·836,539	—	—	—
10·0	7		—	—	—
9·5	1		—	—	—

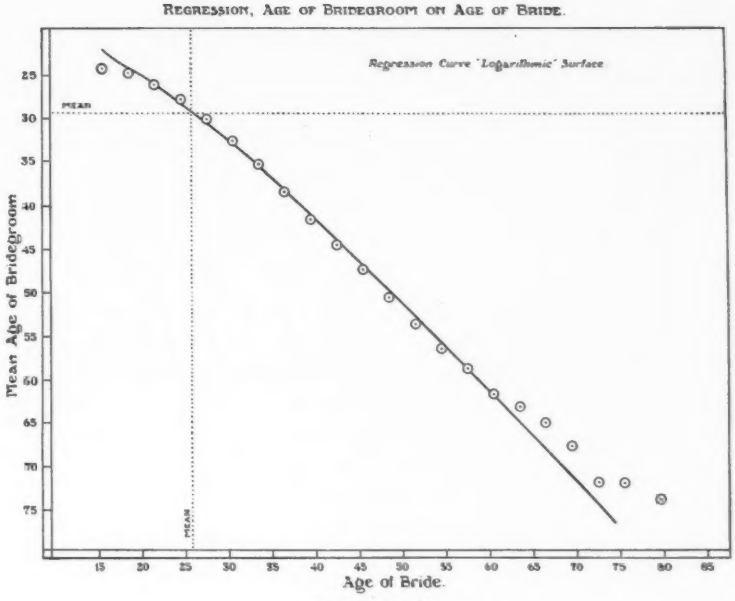
REGRESSION, AGE OF BRIDEGROOM ON AGE OF BRIDE.
Regression Curve 'Logarithmic' Surface.
MEAN
MEAN
Mean Age of Bridegroom
Age of Bride.
15 20 25 30 35 40 45 50 55 60 65 70 75 80 85
25 30 35 40 45 50 55 60 65 70 75

Diagram I (*a*).

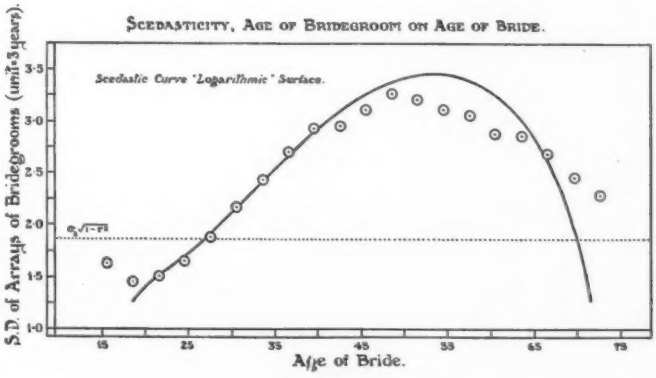
SCEDASTICITY, AGE OF BRIDEGROOM ON AGE OF BRIDE.
Scedastic Curve 'Logarithmic' Surface.
S.D. of Arrays of Bridegrooms (unit 5 years).
Age of Bride.
15 25 35 45 55 65 75
1·0 1·5 2·0 2·5 3·0 3·5

Diagram I (*b*).

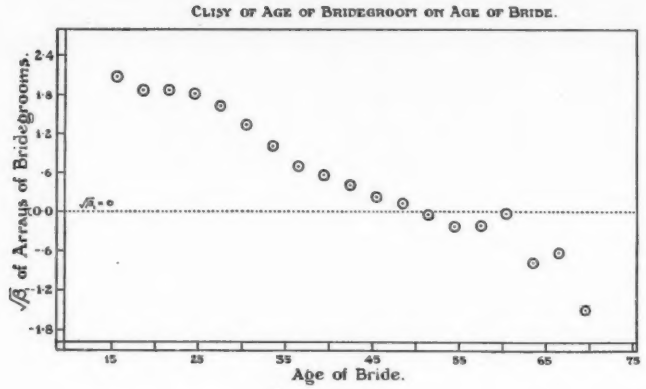

Diagram I (*c*).

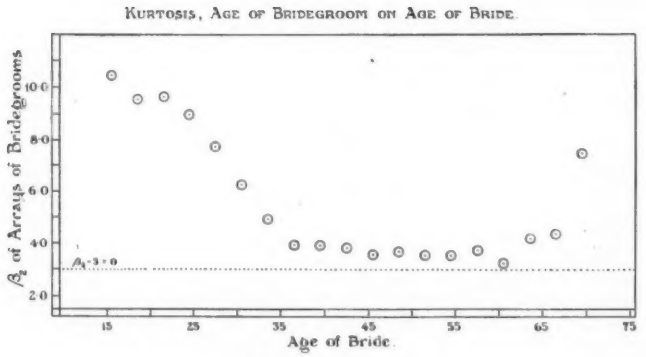

Diagram I (*d*).

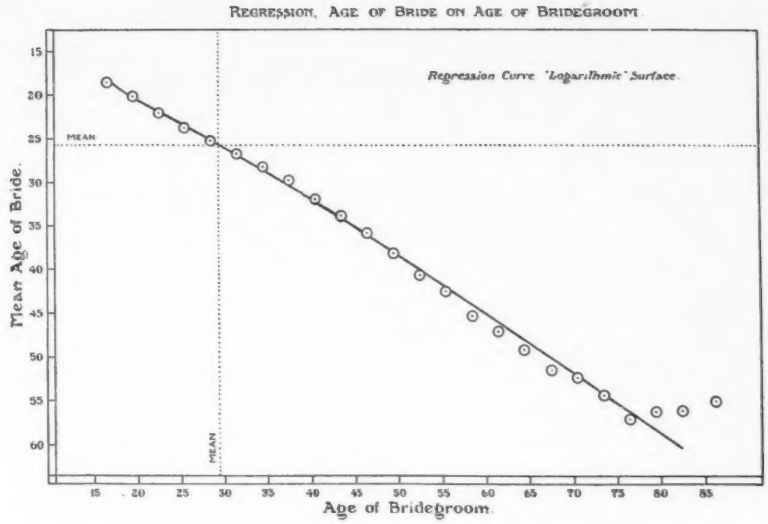

Diagram I (*e*).

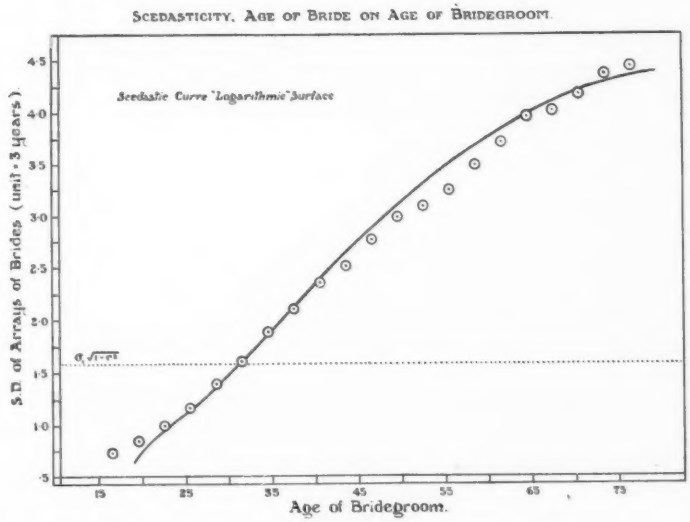
SCEDASTICITY. AGE OF BRIDE ON AGE OF BRIDEGROOM.
Scedastic Curve "Logarithmic" Surface
S.D. of Arrays of Brides (unit = 3 years).
Age of Bridegroom.

Diagram I (*f*).

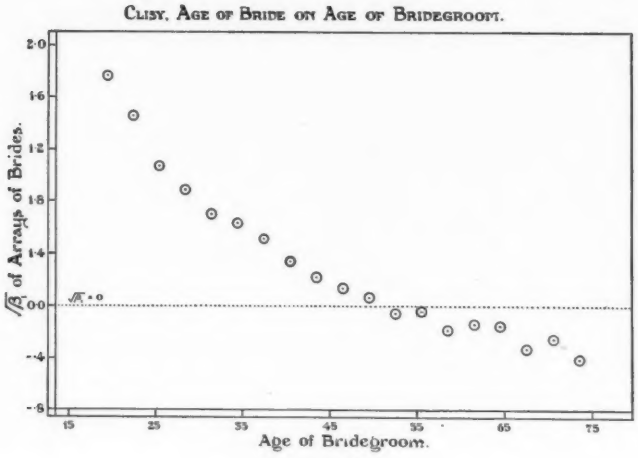

Diagram I (*g*).

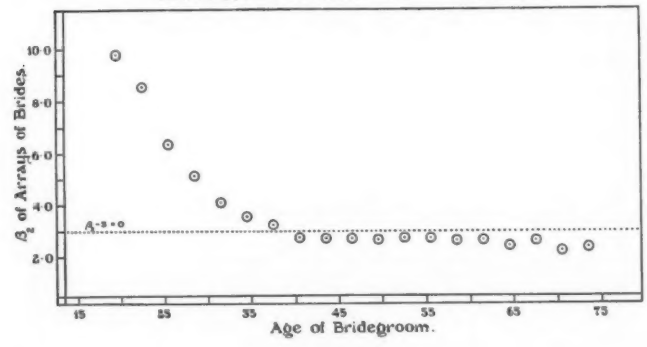

Diagram I (*h*).

REGRESSION, AGE OF FATHER ON AGE OF MOTHER.

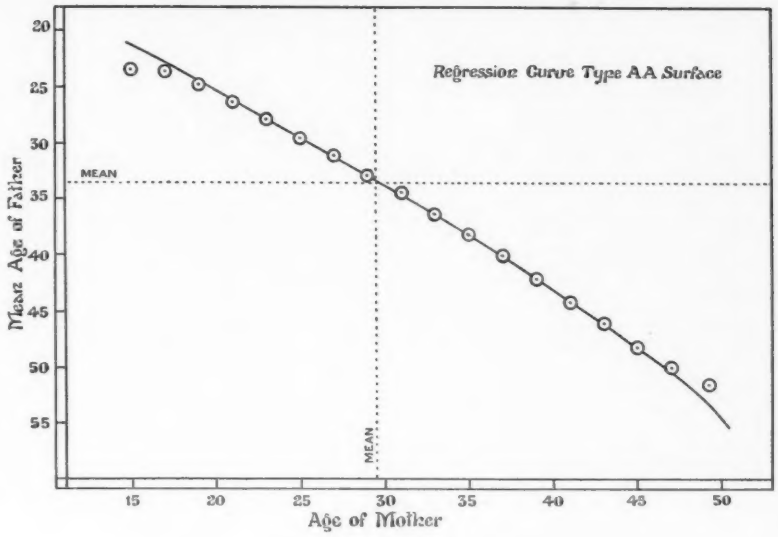

Diagram II (*a*).

SCEDASTICITY. AGE OF FATHER ON AGE OF MOTHER.

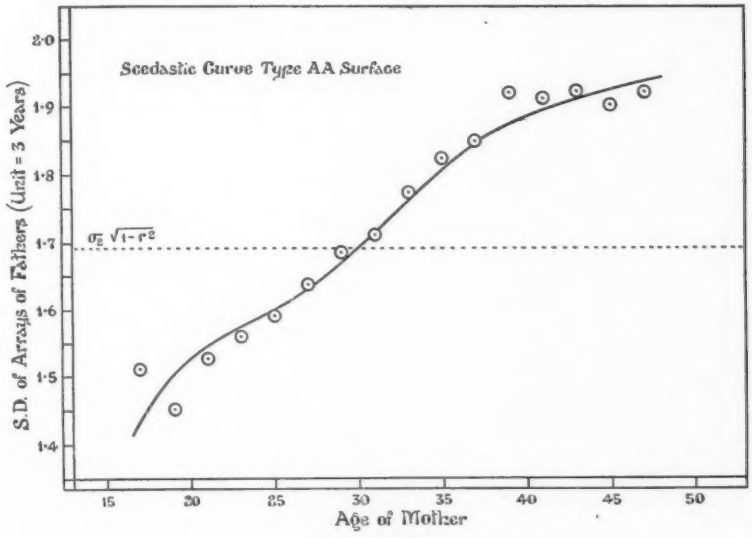

Diagram II (*b*).

GLISY. AGE OF FATHER ON AGE OF MOTHER.

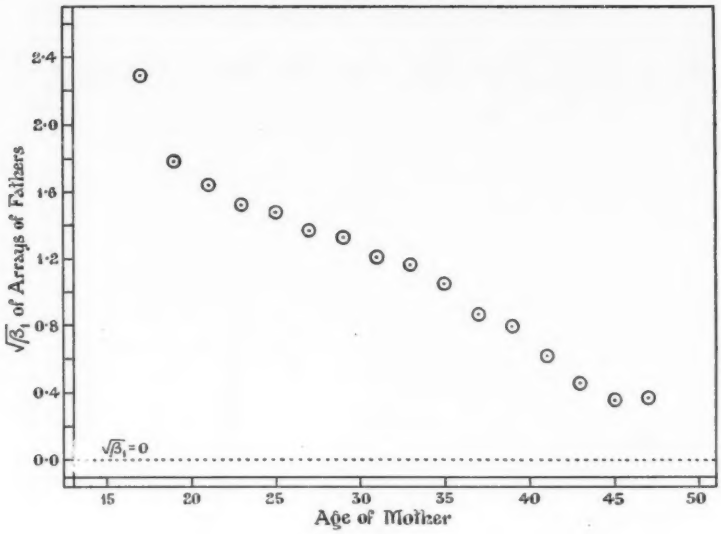

Diagram II (*c*).

KURTOSIS. AGE OF FATHER ON AGE OF MOTHER.

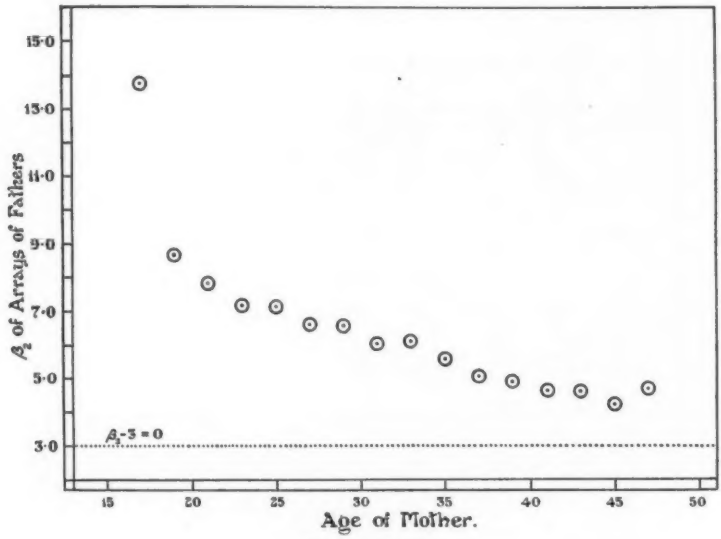

Diagram II (*d*).

REGRESSION, AGE OF MOTHER ON AGE OF FATHER.

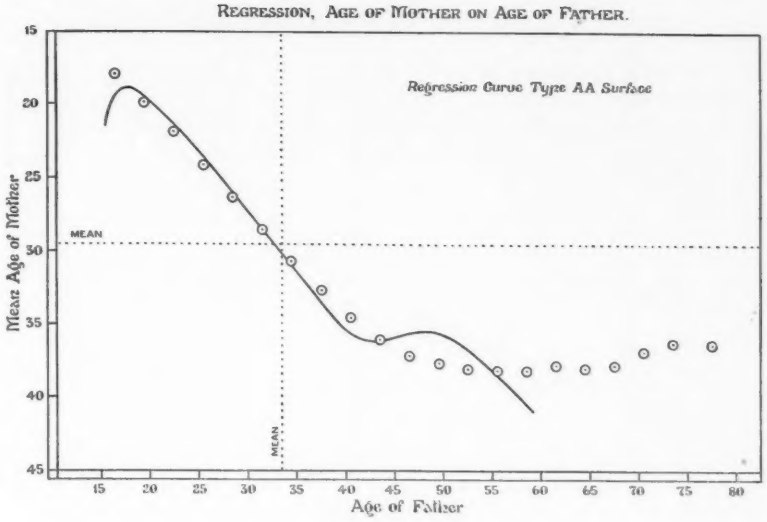

Diagram II (*e*).

SCEDASTICITY, AGE OF MOTHER ON AGE OF FATHER

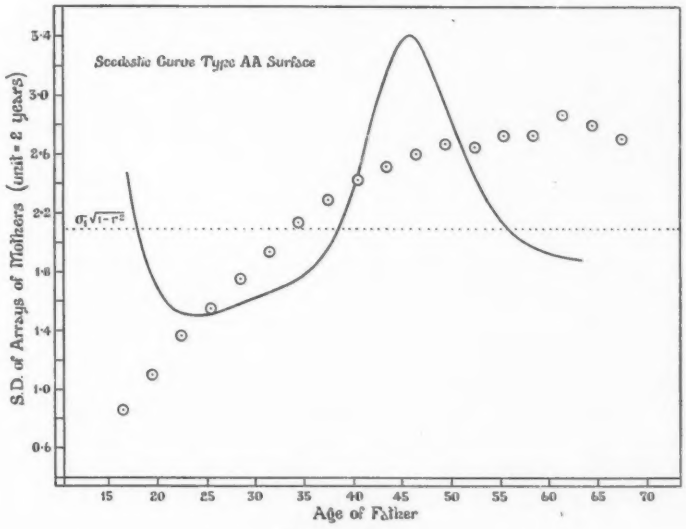

Diagram II (*f*).

GLISY, AGE OF MOTHER ON AGE OF FATHER.

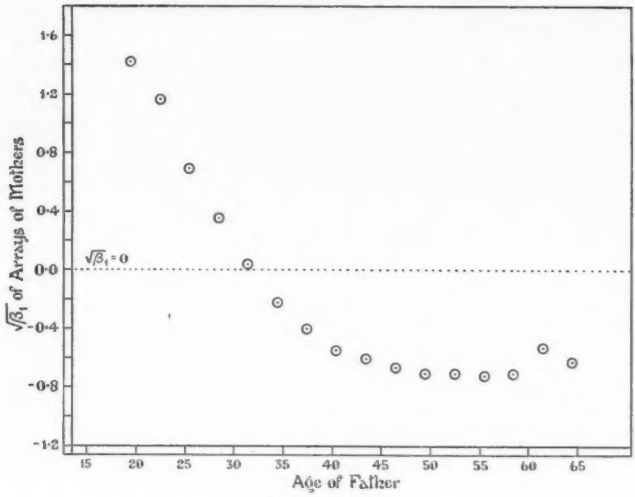

Diagram II (*g*).

KURTOSIS. AGE OF MOTHER ON AGE OF FATHER.

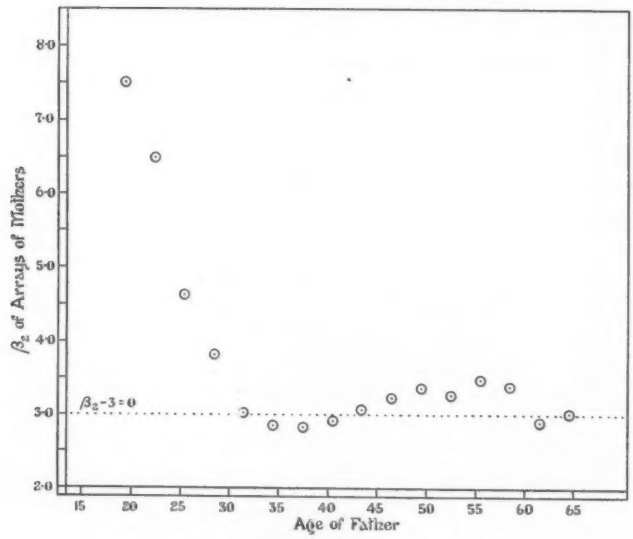

Diagram II (*h*).

REGRESSION, HEIGHT OF BAROMETER: THIRD DAY ON FIRST DAY. (WHOLE YEAR).

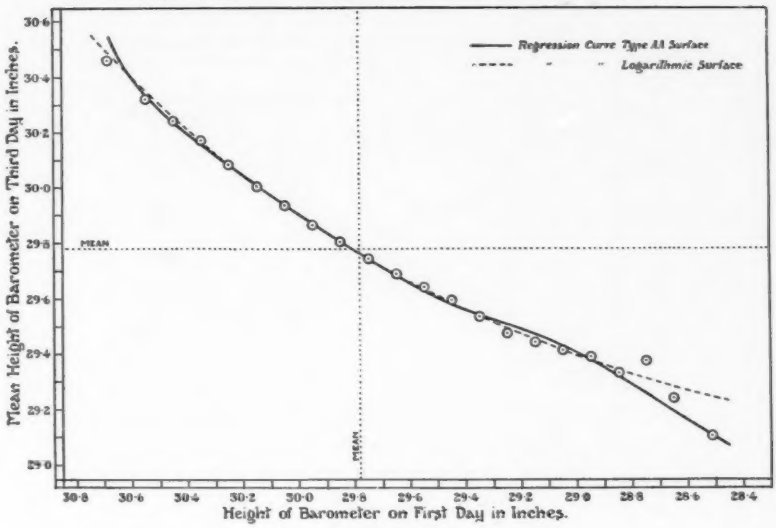

Diagram III (*a*).

SCEDASTICITY, HEIGHT OF BAROMETER: THIRD DAY ON FIRST DAY (WHOLE YEAR).

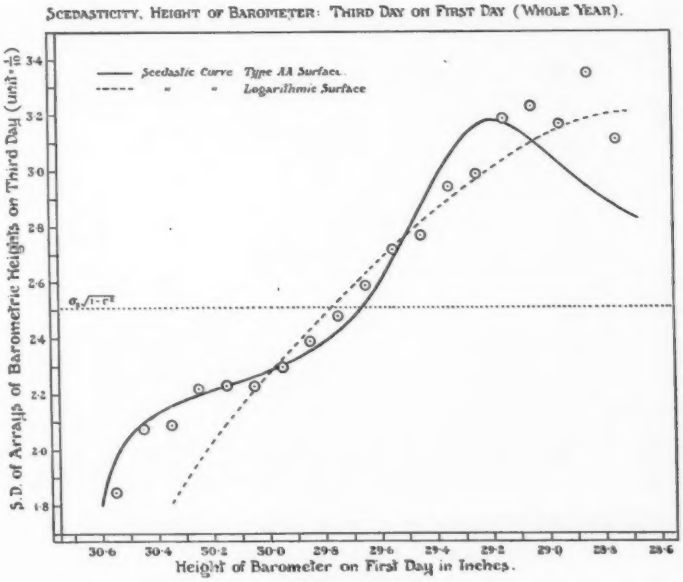

Diagram III (*b*).

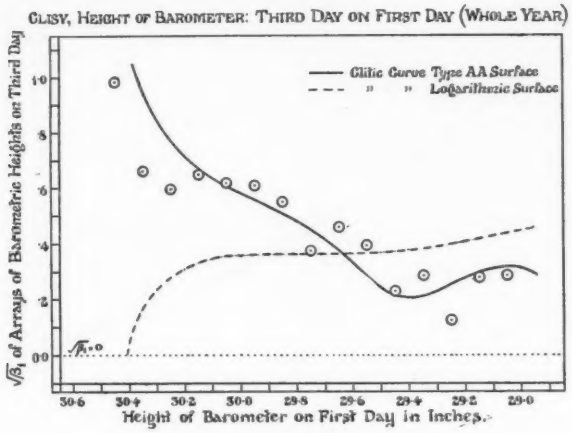
GLISY, HEIGHT OF BAROMETER: THIRD DAY ON FIRST DAY (WHOLE YEAR)
Glific Curve Type AA Surface
" " Logarithmic Surface
$\sqrt{\beta_1}$ of Arrays of Barometric Heights on Third Day
1·0
·8
·6
·4
·2
0·0
$\sqrt{\beta_1}=0$
30·6 30·4 30·2 30·0 29·8 29·6 29·4 29·2 29·0
Height of Barometer on First Day in Inches.

Diagram III (*c*).

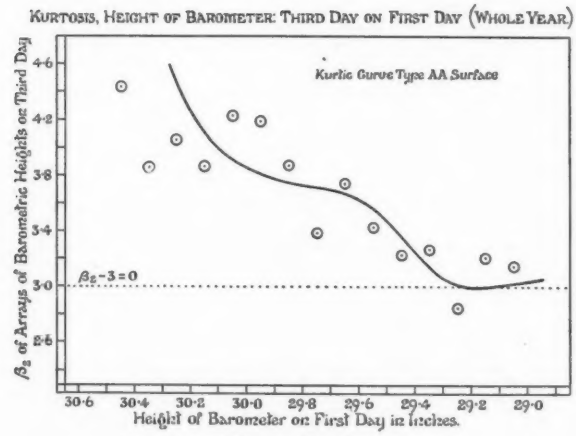
KURTOSIS, HEIGHT OF BAROMETER: THIRD DAY ON FIRST DAY (WHOLE YEAR)
Kurtic Curve Type AA Surface
β_2 of Arrays of Barometric Heights on Third Day
4·6
4·2
3·8
3·4
3·0
2·6
$\beta_2-3=0$
30·6 30·4 30·2 30·0 29·8 29·6 29·4 29·2 29·0
Height of Barometer on First Day in Inches.

Diagram III (*d*).

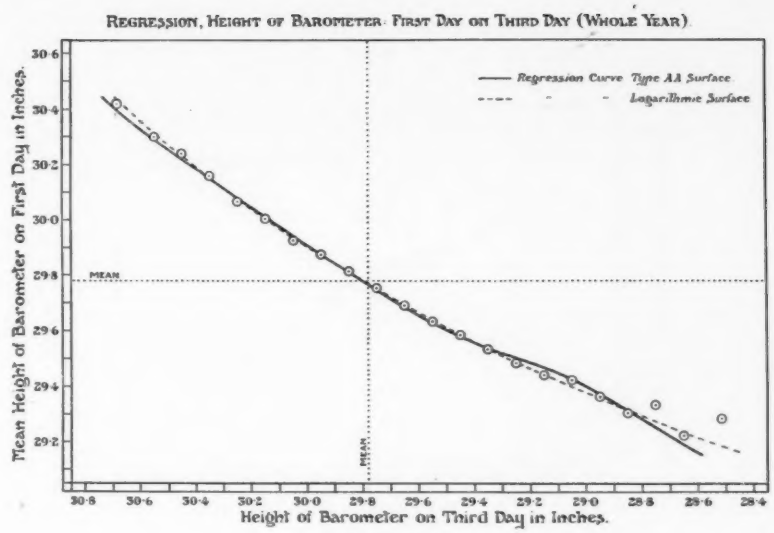
REGRESSION, HEIGHT OF BAROMETER: FIRST DAY ON THIRD DAY (WHOLE YEAR).
Regression Curve Type AA Surface.
Logarithmic Surface.
Mean Height of Barometer on First Day in Inches.
MEAN.
MEAN.
30·6 30·4 30·2 30·0 29·8 29·6 29·4 29·2
30·8 30·6 30·4 30·2 30·0 29·8 29·6 29·4 29·2 29·0 28·8 28·6 28·4
Height of Barometer on Third Day in Inches.

Diagram III (*e*).

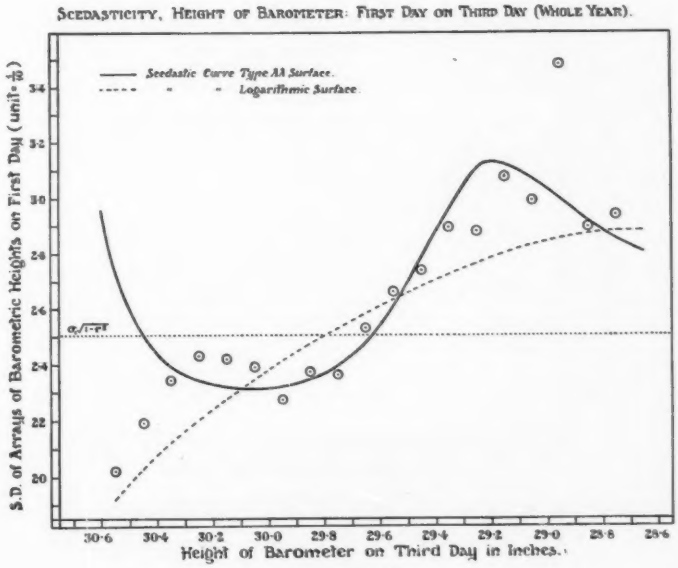
SCEDASTICITY, HEIGHT OF BAROMETER: FIRST DAY ON THIRD DAY (WHOLE YEAR).
Scedastic Curve Type AA Surface.
Logarithmic Surface.
S.D. of Arrays of Barometric Heights on First Day (unit = 1/100)
σ√1−r²
30·6 30·4 30·2 30·0 29·8 29·6 29·4 29·2 29·0 28·8 28·6
Height of Barometer on Third Day in Inches.

Diagram III (*f*).

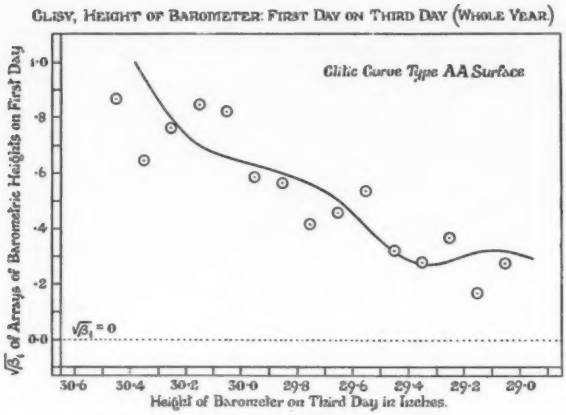
Glisy, Height of Barometer: First Day on Third Day (Whole Year.)
Glitic Curve Type AA Surface
$\sqrt{\beta_1}$ = 0
$\sqrt{\beta_1}$ of Arrays of Barometric Heights on First Day
1·0
·8
·6
·4
·2
0·0
30·6 30·4 30·2 30·0 29·8 29·6 29·4 29·2 29·0
Height of Barometer on Third Day in Inches.

Diagram III (*g*).

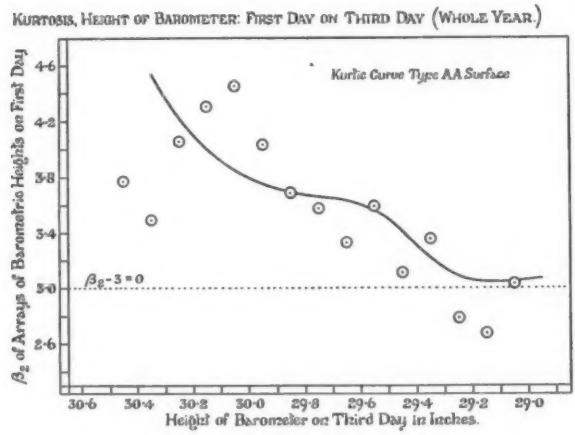
Kurtosis, Height of Barometer: First Day on Third Day (Whole Year.)
Kurtic Curve Type AA Surface
$\beta_2 - 3 = 0$
β_2 of Arrays of Barometric Heights on First Day
4·6
4·2
3·8
3·4
3·0
2·6
30·6 30·4 30·2 30·0 29·8 29·6 29·4 29·2 29·0
Height of Barometer on Third Day in Inches.

Diagram III (*h*).

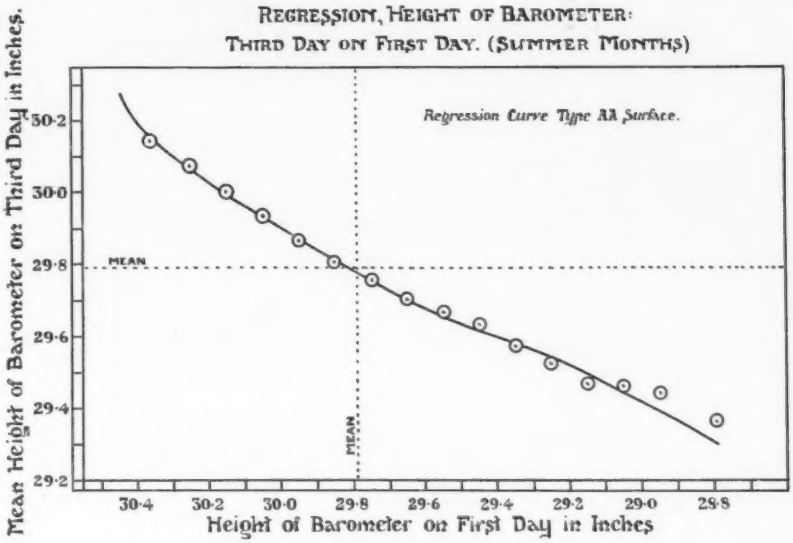

Diagram IV (*a*).

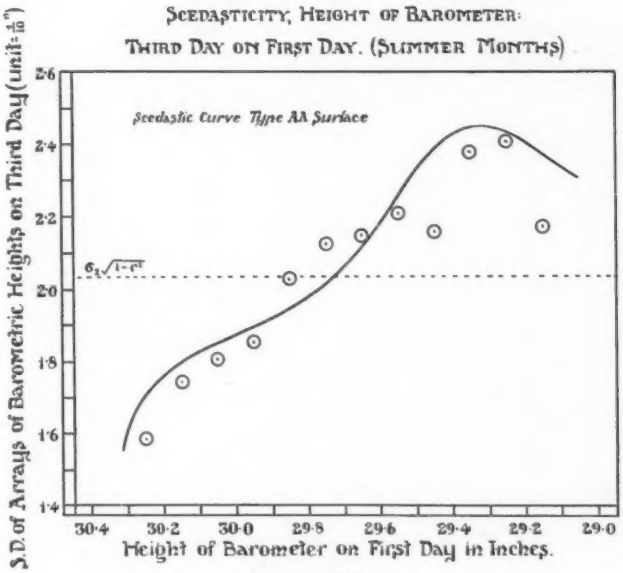

Diagram IV (*b*).

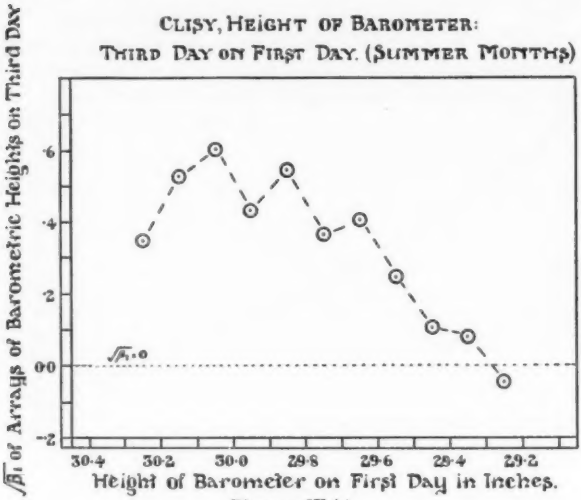

Diagram IV (*c*).

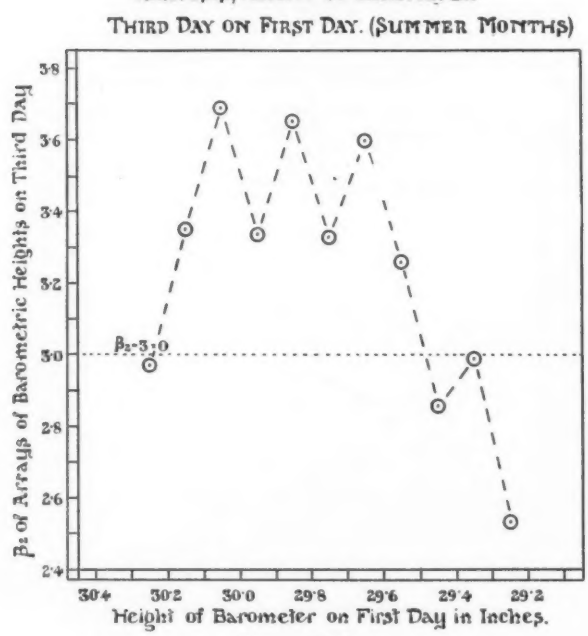

Diagram IV (*d*).

REGRESSION, HEIGHT OF BAROMETER:
FIRST DAY ON THIRD DAY (SUMMER MONTHS)

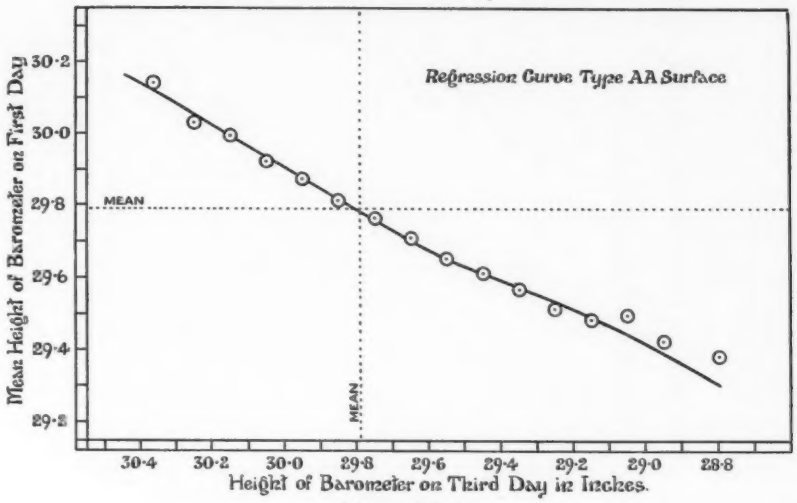

Diagram IV (*e*).

SCEDASTICITY, HEIGHT OF BAROMETER:
FIRST DAY ON THIRD DAY (SUMMER MONTHS)

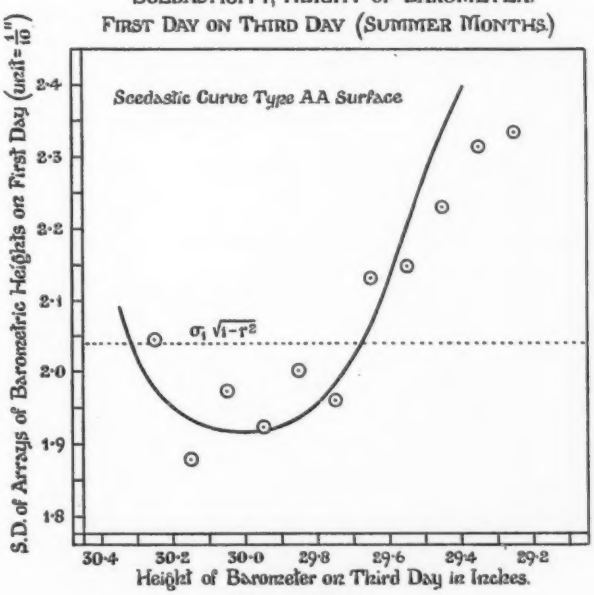

Diagram IV (*f*).

GLISY, HEIGHT OF BAROMETER.
FIRST DAY ON THIRD DAY (SUMMER MONTHS.)

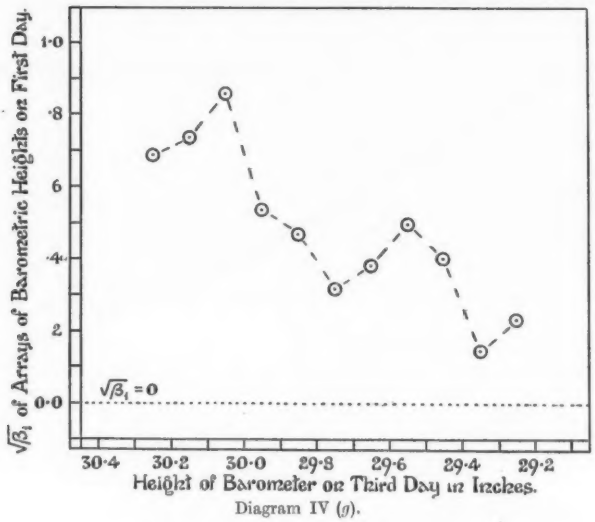

Diagram IV (*g*).

KURTOSIS, HEIGHT OF BAROMETER:
FIRST DAY ON THIRD DAY (SUMMER MONTHS)

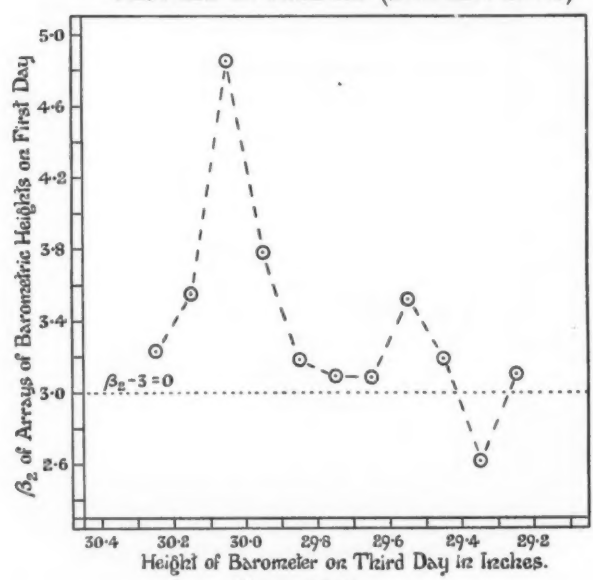

Diagram IV (*h*).

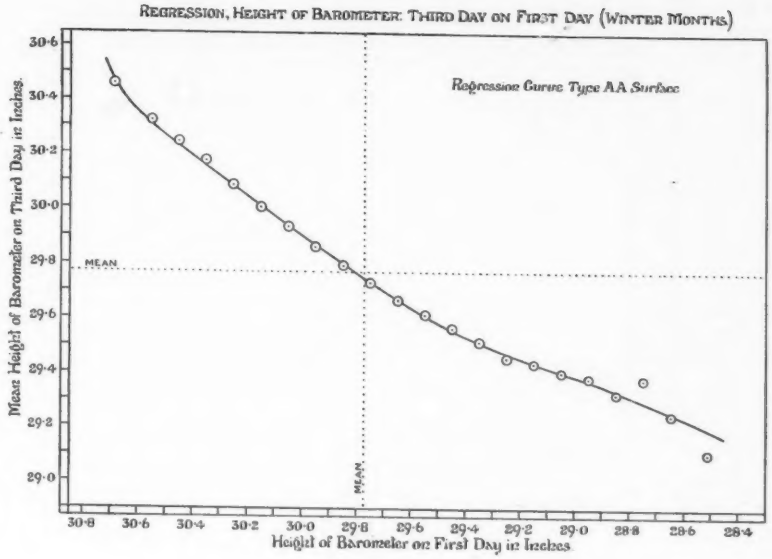

Diagram V (*a*).

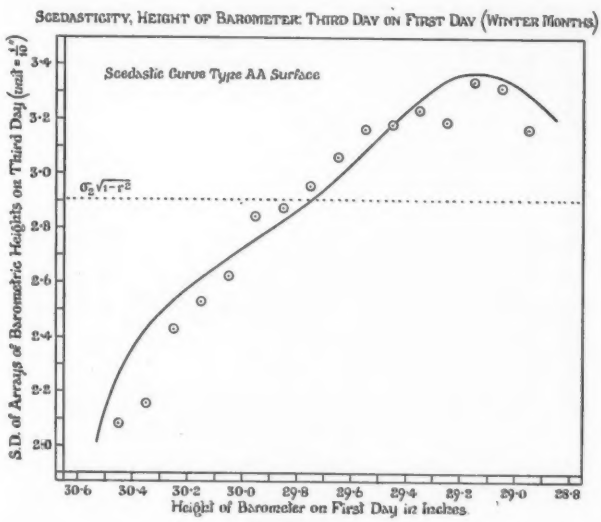

Diagram V (*b*).

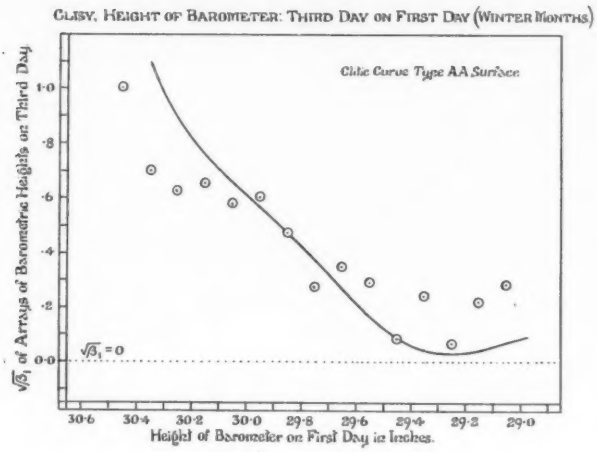
CLISV. HEIGHT OF BAROMETER: THIRD DAY ON FIRST DAY (WINTER MONTHS)
Cubic Curve Type AA Surface
$\sqrt{\beta_1}$ = 0
$\sqrt{\beta_1}$ of Arrays of Barometric Heights on Third Day
1·0
·8
·6
·4
·2
0·0
30·6 30·4 30·2 30·0 29·8 29·6 29·4 29·2 29·0
Height of Barometer on First Day in Inches.

Diagram V (*c*).

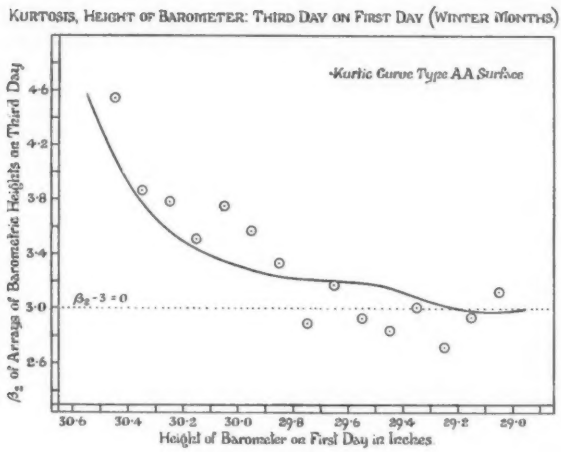
KURTOSIS, HEIGHT OF BAROMETER: THIRD DAY ON FIRST DAY (WINTER MONTHS)
Kurtic Curve Type AA Surface
$\beta_2 - 3 = 0$
β_2 of Arrays of Barometric Heights on Third Day
4·6
4·2
3·8
3·4
3·0
2·6
30·6 30·4 30·2 30·0 29·8 29·6 29·4 29·2 29·0
Height of Barometer on First Day in Inches.

Diagram V (*d*).

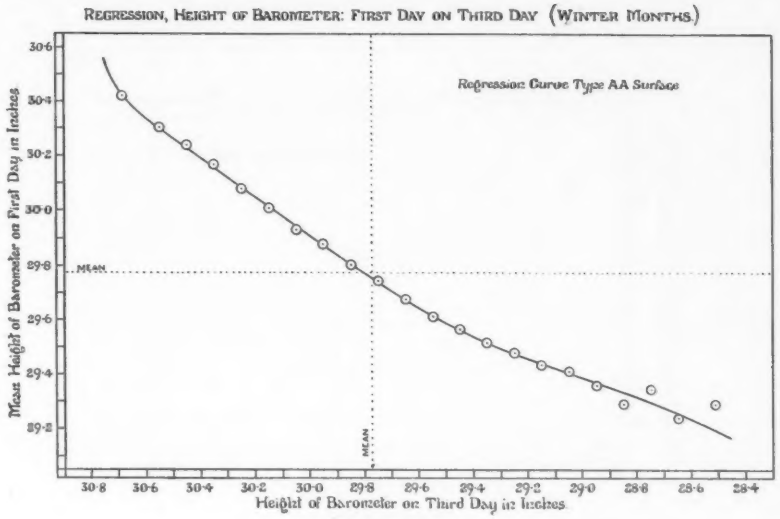

Diagram V (*c*).

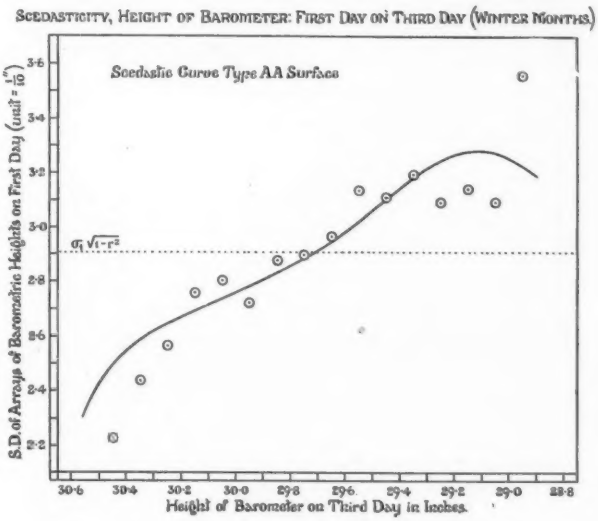

Diagram V (*f*).

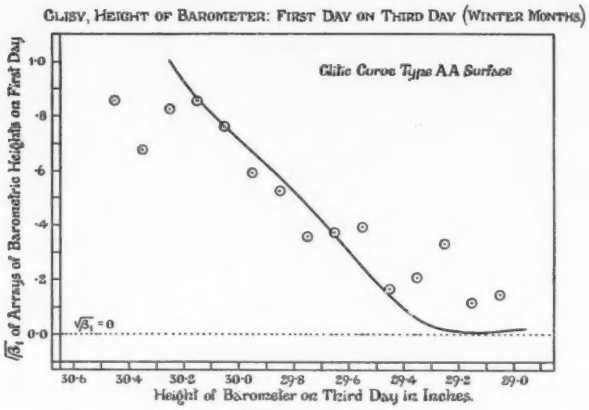

Diagram V (*g*).

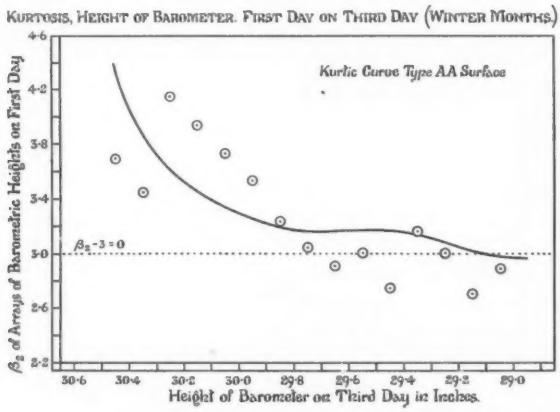

Diagram V (*h*).

REGRESSION, BREADTH ON LENGTH OF BEANS.

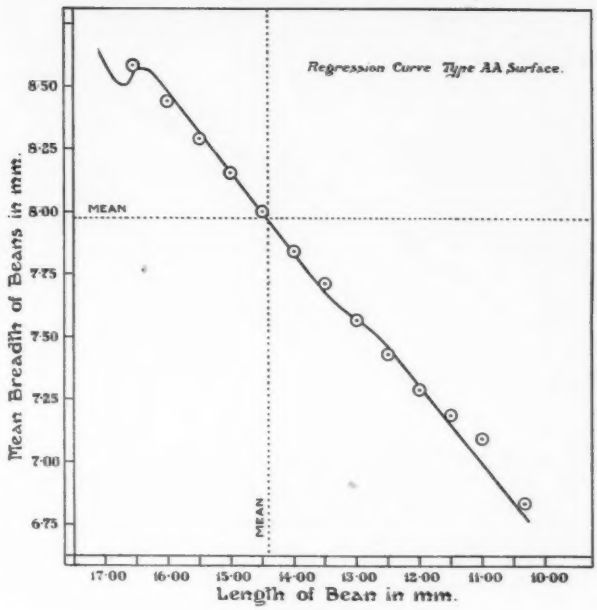

Diagram VI (*a*).

SCEDASTICITY. BREADTH ON LENGTH OF BEANS.

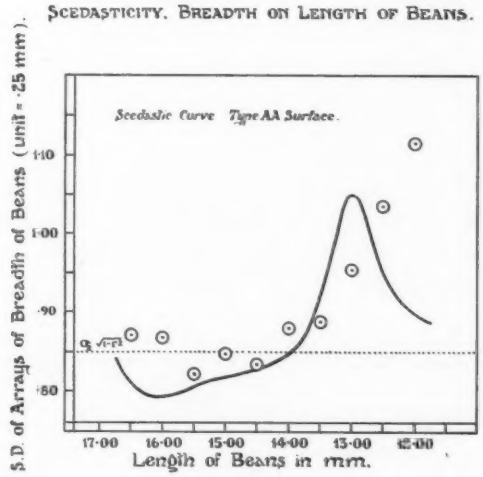

Diagram VI (*b*).

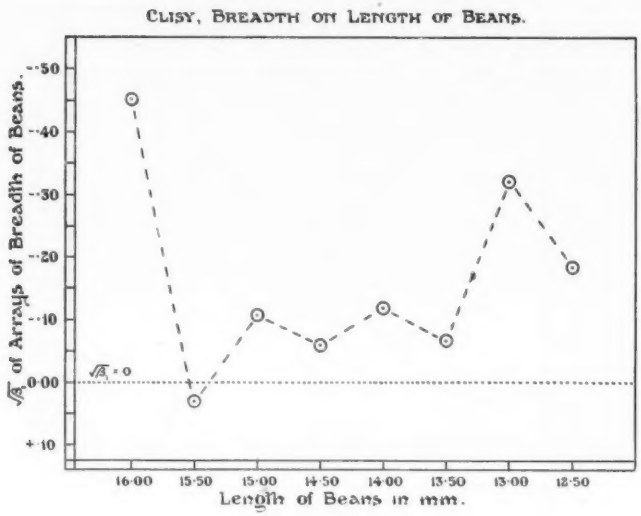

Diagram VI (*c*).

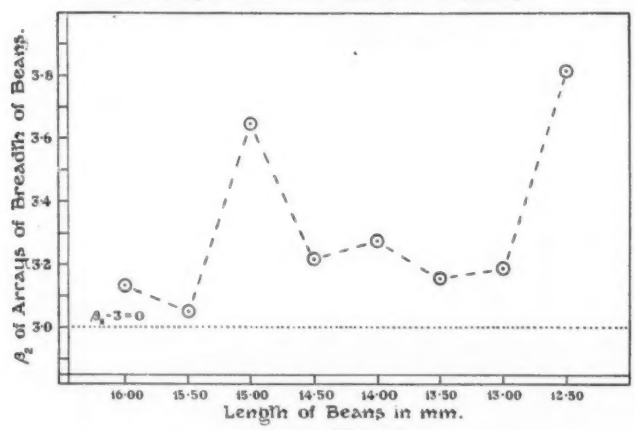

Diagram VI (*d*).

REGRESSION, LENGTH ON BREADTH OF BEANS.

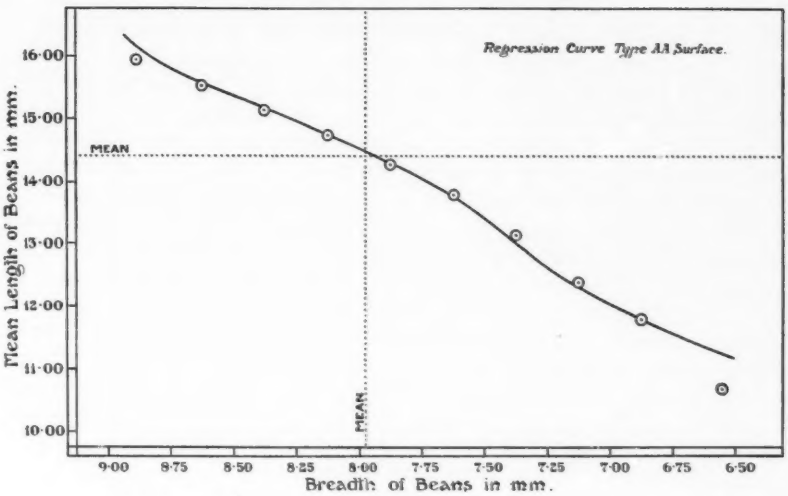

Diagram VI (*e*).

SCEDASTICITY, LENGTH ON BREADTH OF BEANS.

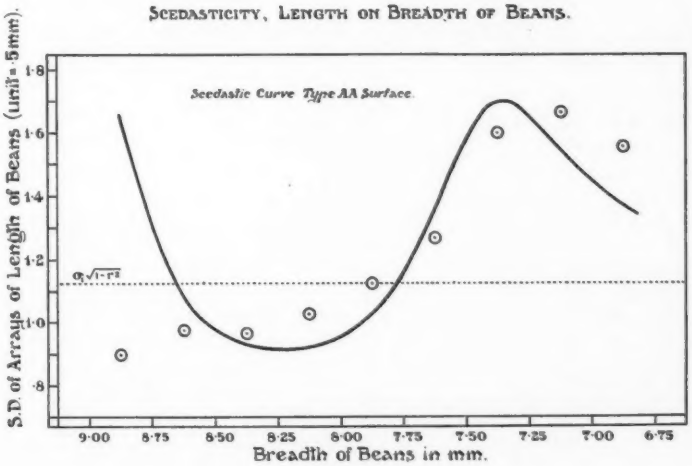

Diagram VI (*f*).

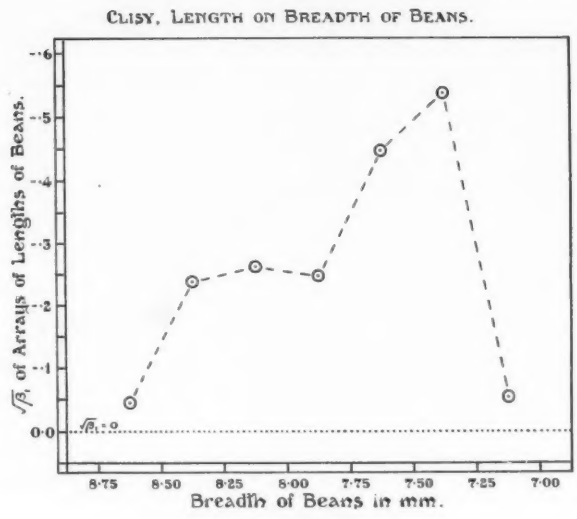

Diagram VI (*g*).

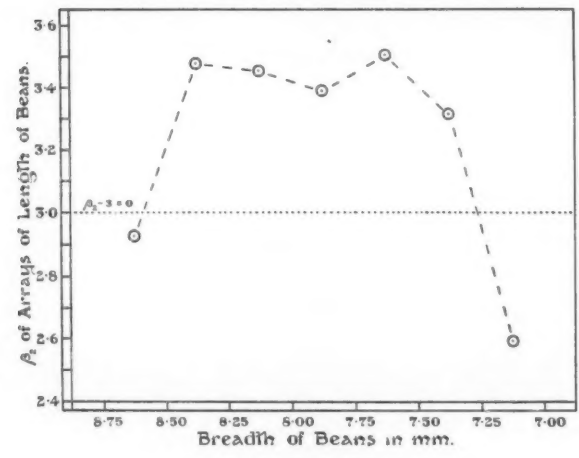

Diagram VI (*h*).

abruptness will be dealt with in Section 5; as a matter of fact, they were found to be negligible.

The observation points are represented by small circles in Diagrams I(*a*), I(*b*), up to VI(*h*). Except at the extremities of the distributions, the means follow fairly smooth trends. The regression of age of mother on age of father is of the rectangular hyperbolic type, while the other regressions deviate less from rectilinearity. The scedasticity reveals, not infrequently, a resemblance to the parabolic form of the double hypergeometrical series; for example, in the barometric and birth statistics. Approaching the higher moments we find, as could have been anticipated, that the observation points become more and more erratic. Yet, for the first two distributions, the variation in the clisy and kurtosis is still remarkably regular; for the barometric data a quite definite trend, other than constant, is noticeable; and it is only for the distribution of length and breadth of beans, where both the number of points on the graphs and the total number of observations are small, that these two measures seem to be scattered at random. This finding of a fairly regular variation in the shape of the array distributions disproves, beyond doubt, the generality of Narumi's hypothesis for these cases.

We proceed to fit to these points the appropriate curves of the Type AaAa* and of the Logarithmic Surface. The expressions for the partial moment curves of the former surface are given in equations (62), (63), (64), (65), and (66); those for the Logarithmic Surface in equations (39) and (40). The constants in these formulae together with the values they assume for the different distributions are shown in Tables VII and VIII. The surfaces will be considered in the order named.

Both regression curves of the Type AaAa surface were fitted to the array means of the last five distributions. The curves are geometrically somewhat quaint, but, with the exception of the regression of age of mother on age of father, the results are quite reasonable. The scedastic curves fitted for the same distributions are oscillatory to a much higher degree than the regression curves; even in the most favourable cases the fit is rather unsatisfactory. The clitic and kurtic curves were fitted for the barometric data: whole year and winter months. For the first of these distributions they do not fit badly: perhaps they fit even better than the corresponding scedastic curves do. For the latter distribution, however, they are less successful.

The regression curves of the Logarithmic Surface were tested for the barometric (whole year) and marriage distributions†. They fit very well in the former case and probably better than the curves of the Type AaAa surface do. For the marriage distribution a want of flexibility in the curves is manifest. This inefficiency is still more marked when we come to the scedastic curves. For the barometric distribution these curves, bad as they are, are to be preferred to those of the Type AaAa surface; for the marriage distribution, too, the results are

* Read *Type AaAa* instead of *Type AA* in the diagrams.

† See Section 4 for a discussion of the criteria justifying the application of this surface to these two distributions.

TABLE VII.

Coefficients of Tetrachoric Terms in Expressions for Partial Moment Curves of Type Aa Aa Surface.

		a_4	a_5	b_3	b_4	c_2	c_3	d_1	d_2	e_1	$1-r^2$
Australian Marriages	y_x	—	—	—	—	—	—	—	—	—	—
	x_y	—	—	—	—	—	—	—	—	—	
Australian Births	y_x	+·258,976	−·260,019	+·051,479	+·030,464	−·097,736	−·011,932	—	—	—	·459,857
	x_y	+·591,414	+·284,893	−·281,848	−·488,514	−·314,184	−·366,421	—	—	—	
Barometer Whole Year	y_x	+·368,070	+·181,553	−·134,386	−·074,911	−·204,774	−·140,828	+·221,757	−·087,997	+·337,295	·662,763
	x_y	+·368,476	+·182,021	−·113,524	−·088,514	−·152,062	−·164,106	+·268,864	−·088,545	+·311,655	
Barometer Summer Months	y_x	+·366,599	+·125,252	−·107,846	−·051,409	−·190,742	−·082,283	—	—	—	·714,262
	x_y	+·352,902	+·098,120	−·075,839	−·071,878	−·133,854	−·135,910	—	—	—	
Barometer Winter Months	y_x	+·307,898	−·063,206	−·150,667	−·090,200	−·185,351	−·016,549	+·193,837	−·194,781	+·133,561	·638,846
	x_y	+·309,196	−·062,860	−·132,495	−·098,180	−·136,986	−·030,599	+·237,484	−·111,105	+·118,593	
Beans	y_x	−·743,476	+·850,314	+·071,055	−·121,665	+·053,061	−·052,302	—	—	—	·389,818
	x_y	−·359,938	+·298,680	−·193,685	+·178,030	+·194,051	−·253,177	—	—	—	

rather poor. Finally the clitic curve (y on x), fitted to the barometric data, tends to a constant value.

In considering the goodness of fit of these curves we must bear in mind, as Pearson pointed out when he discussed the 15-Constant surface, that the constants in the equations of the curves are determined not directly from the moments of

TABLE VIII.

Constants in Expressions for Partial Moment Curves of Logarithmic Surface.*

		Australian Marriages		Barometer Whole Year	
		y on x	x on y	y on x	x on y
Margins	ξ	3·73516	4·49910	20·64737	20·62685
	e	·50562	·58884	1·31009	1·30965
	s	·24068	·23630	·06442	·06449
Regression	$\lambda^{(1)}$	1·440,318	1·494,276	1·365,984	1·363,189
	$\gamma^{(1)}$	·788,336	·942,229	1·793,429	1·789,158
	$d^{(1)}$	·589,051	·672,275	1·315,756	1·315,314
	$D_0^{(1)}$	·942,228	·955,639	1·005,604	1·003,222
	$D_1^{(1)}$	·098,457	·138,009	− ·007,433	− ·012,877
	$D_2^{(1)}$	·997,329	·794,485	−1·350,278	− ·774,629
Scedasticity	$\lambda^{(2)}$	2·880,636	2·988,552	2·731,967	2·726,378
	$\gamma^{(2)}$	1·696,844	2·009,130	3·594,603	3·586,043
	$d^{(2)}$	·672,485	·755,709	1·321,425	1·320,983
	$D_0^{(2)}$	·884,455	·911,279	1·011,208	1·006,443
	$D_1^{(2)}$	·393,826	·552,034	− ·029,732	− ·051,509
	$D_2^{(2)}$	1·994,657	1·588,971	−2·700,557	−1·549,257
Cliey	$\lambda^{(3)}$	—	—	4·097,951	—
	$\gamma^{(3)}$	—	—	5·403,520	—
	$d^{(3)}$	—	—	1·327,094	—
	$D_0^{(3)}$	—	—	1·016,812	—
	$D_1^{(3)}$	—	—	− ·066,896	—
	$D_2^{(3)}$	—	—	−4·050,835	—

the array distributions, but from the momental constants used in fitting the surface as a whole. The fit of the regression curves, excluding the regression of age of mother on age of father, does not leave much to be desired. The practical utility of the curves, however, is greatly reduced by their algebraic complexity. For the higher moment curves, generally speaking, the results are rather unsatisfactory. How the flexibility of a surface is affected by the number of constants in the equation is shown clearly by Diagrams III (a), III (b) and III (c); two terms

* For the marriage data $B_{21} = -\cdot001,453$, $B_{12} = -\cdot001,076$, $\rho = \cdot637,131$ found from equations (36). For the barometric data $B_{21} = +\cdot000,0101$, $B_{12} = +\cdot000,0058$, $\rho = \cdot592,632$ found from equations (35) and (37).

added to the equation of the Logarithmic Normal Surface are not enough to effect an arbitrary degree of variability in the constant clitic curve of this surface.

3. *Contours.* We turn now to the contours, or curves of equal probability, of the data to study not only their form, but also their symmetry about a set of principal axes. They were constructed directly from the observed frequencies. The labour involved in replacing the frequencies by ordinates would have been prohibitive; besides, by such a treatment of frequencies negative ordinates appear in the tails of the arrays. The method adopted in determining the contour lines is the same as that described by Pearson in *Biometrika*, Vol. XVII. pp. 296–97. The smoothing of the array curves by aid of a spline, the plotting of the contours from these curves, the smoothing of the contours and their final adjustment to

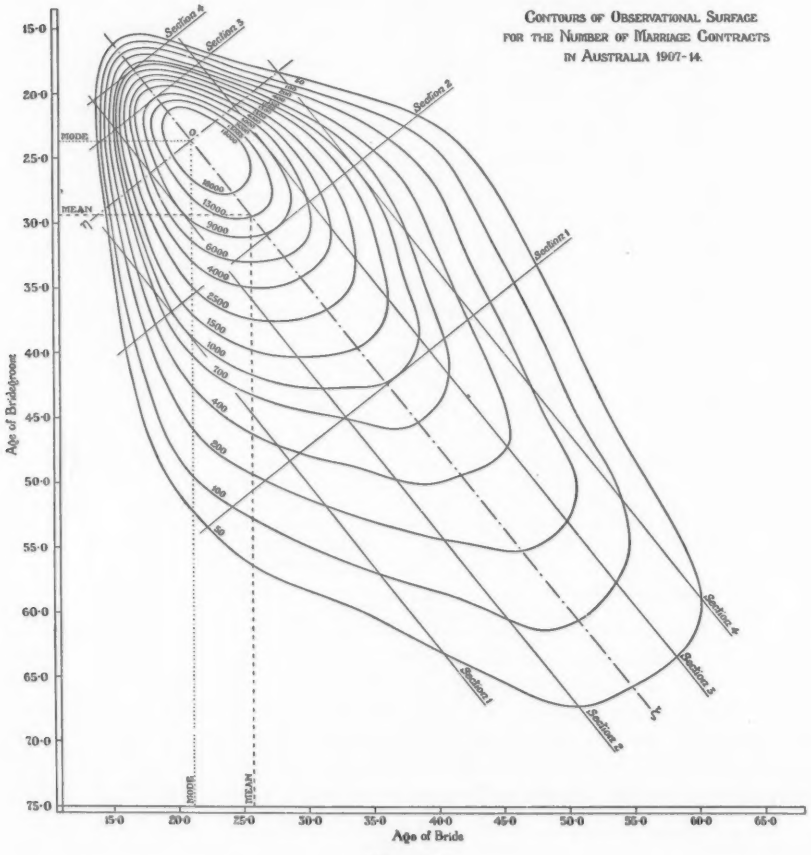

Diagram I.

one another, are the main steps in the process. Each of these steps called for much painstaking in preventing the personal equation from playing too great a part. The final solutions are shown in Diagrams I to VI.

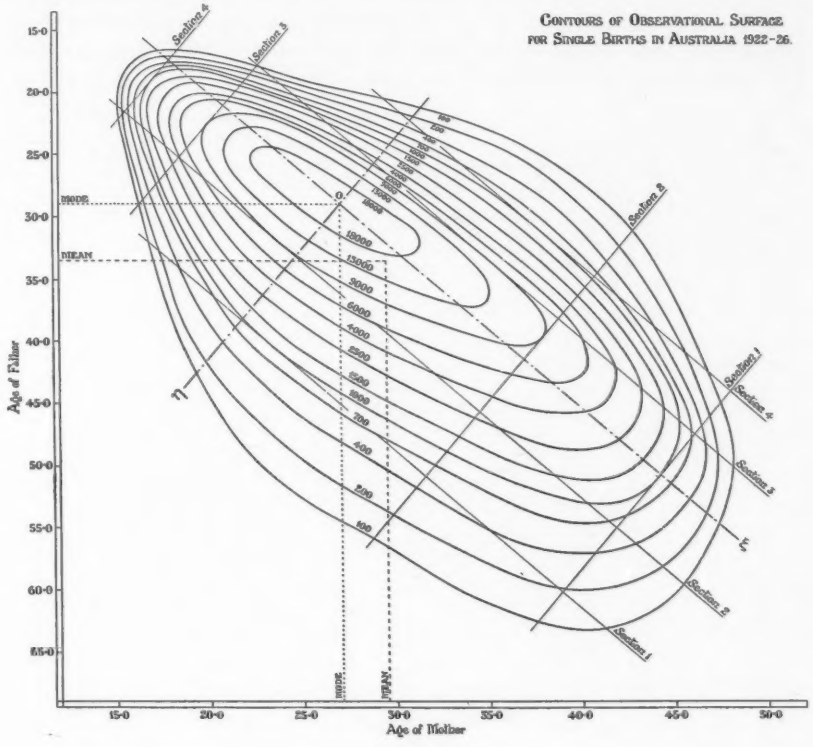

Diagram II.

In form, the contours occasionally resemble the ovals inside a Bernoulli's lemniscate; for instance, those of the three barometric height distributions. The distribution of length and breadth of beans indicates a system of ellipses not concentrically placed; for the contours of the other two distributions no definite laws are recognisable. In fact, it is very doubtful whether any mathematical surface could reproduce the asymmetry displayed in the distribution of ages of parents at births of children.

The idea of axes of independent probability is not applicable, as is fairly evident from the diagrams, to any of the distributions. By considering a set of principal axes such as $O\xi$, $O\eta$—to be described later on—we can easily convince ourselves that whereas sections parallel to the major axis might be homoscedastic and similar, sections parallel to the minor axis, if not dissimilar, are heteroscedastic.

If these sections, reduced to homoscedasticity, were found to be similar, an extension of Narumi's hypothesis would be justified.

The set of principal axes which would be the most likely to give similar parallel sections, was that through the modes of the distributions. Accordingly it was first of all necessary to locate the positions of the modes as accurately as possible. This was done by the one or the other of the two following methods according to the skewness of the distribution. Both methods depend on the modes of the

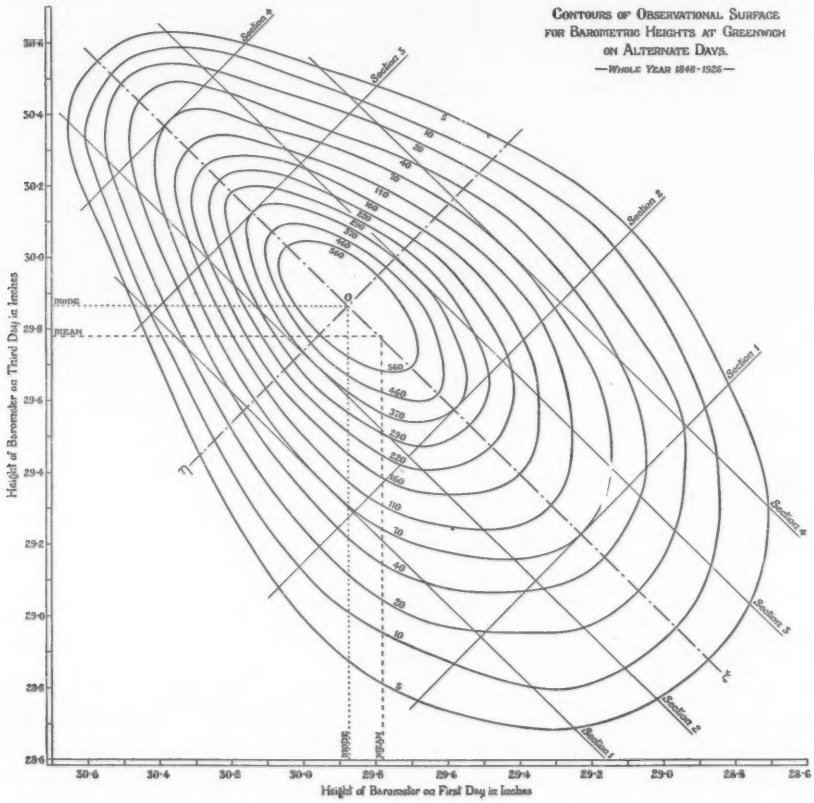

Diagram III.

arrays, centred about the mode of the surface, first being determined. Usually four rows and four columns were found to be adequate. The intersection of the curves—either straight lines or parabolas of the second or higher order—fitted to the modes of these arrays was taken to be the mode of the surface. The modes of the arrays were determined either from the β-formula expressing the distance between the mean and mode for the Pearsonian system of curves, or, by fitting a cubic by the method of least squares to the six largest frequencies in the separate arrays.

The latter method was used for the first two distributions; the former for the last four. Through the modes thus located the sets of principal axes $O\xi$, $O\eta$ were constructed. The following table gives the positions of the modes, and the angle (θ) $O\xi$ makes with the x-axis.

Table I	x mode = 21·20 years y mode = 23·70 „ $\theta = 51° 33'$	Table IV	x mode = 29·86″ y mode = 29·86″ $\theta = 44° 28'$
Table II	x mode = 27·08 years y mode = 28·95 „ $\theta = 40° 19'$	Table V	x mode = 29·96″ y mode = 29·95″ $\theta = 44° 29'$
Table III	x mode = 29·87″ y mode = 29·86″ $\theta = 44° 36'$	Table VI	x mode = 14·62 mm. y mode = 8·04 mm. $\theta = 34° 48'$

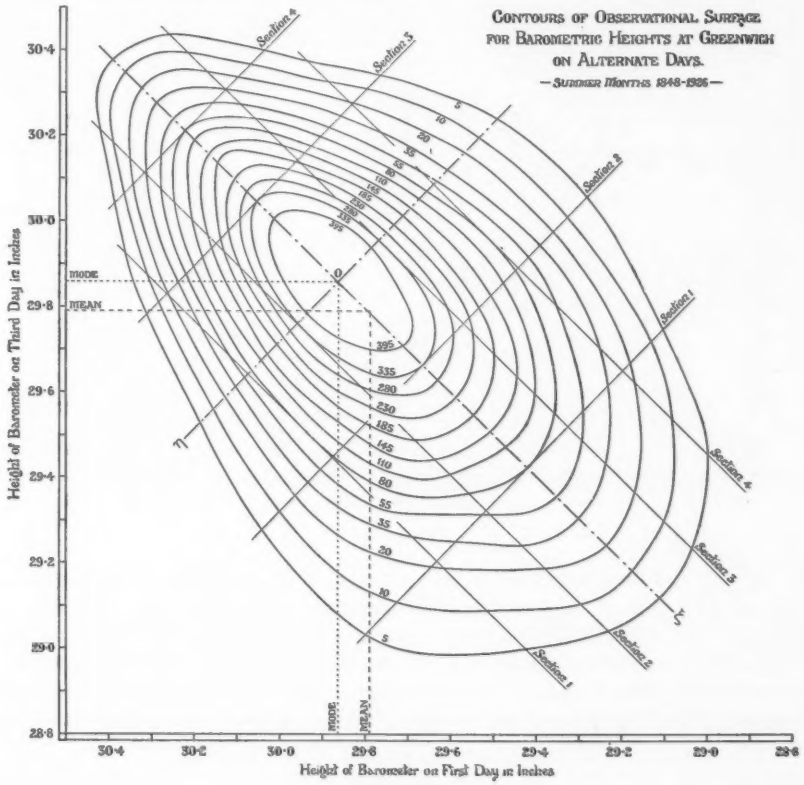

Diagram IV.

There is one interesting feature about the axes that might be pointed out. In five out of the six cases the major axis ($O\xi$) passes, if not completely then very nearly, through the observed mean. Accordingly, we could almost just as well have considered the principal axes through the mean instead of those through the mode.

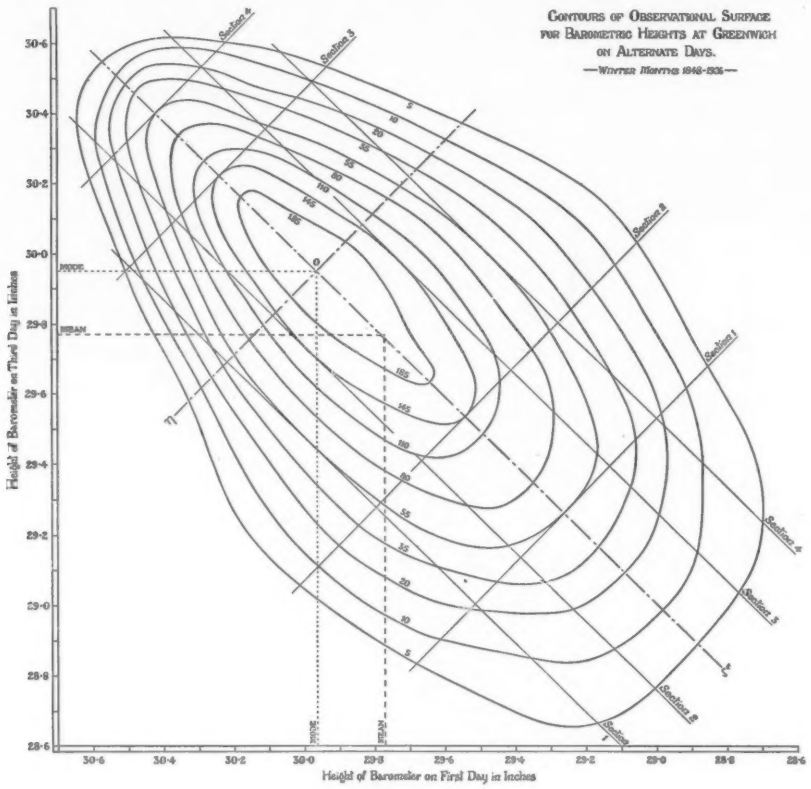

Diagram V.

We proceed to test sections parallel to the axes for similarity. Eight sections, two on each side of an axis, were examined for each of the first five contour systems. The cuts were made in such a way that they touched two of the contour lines. This at once fixed the modal ordinates and defined the vertical scale of the sections. The tails of the section-curves were determined from the array-curves originally used for plotting the contours. The area under each curve was subdivided and planimetered, whence the first two moments were computed. Next, the standard deviations and modal ordinates of each set of four sections were

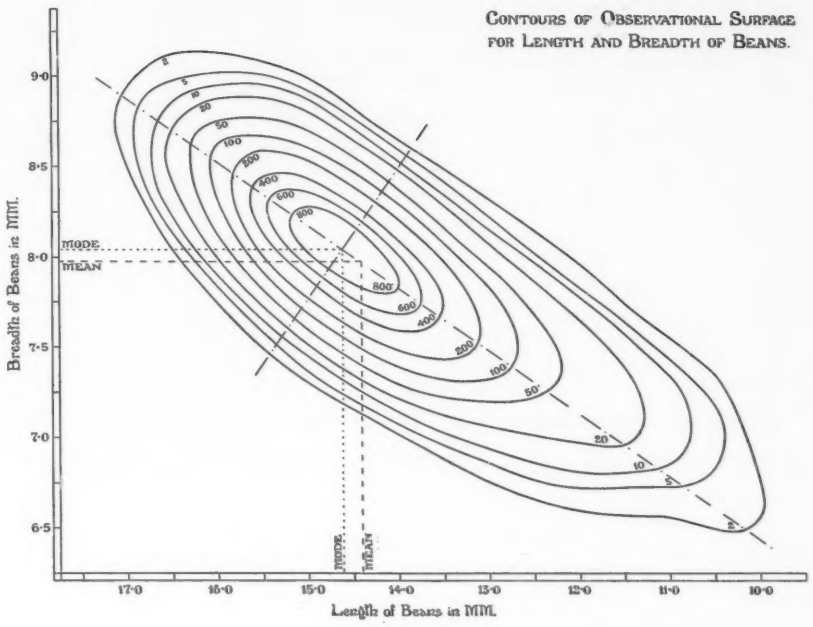

Diagram VI.

equalised, and the two sections differing most were superposed. Diagrams I (i), I (j) to V (i), V (j) show the results. As an illustration, I take the distribution of barometric heights (whole year). The modal ordinate, standard deviation, and skewness (Sk.) $= \dfrac{\text{mean} - \text{mode}}{\text{standard deviation}}$ of the sections are given below.

	Sections parallel to $O\xi$				Sections parallel to $O\eta$			
	Section 1	Section 2	Section 3	Section 4	Section 1	Section 2	Section 3	Section 4
Mod. Ord.	70	370	370	70	70	370	370	70
S. D.*	3·77	3·79	3·81	3·79	2·74	2·26	1·52	1·40
Sk.	+·11	+·20	+·17	+·24	+·02	+·06	−·02	+·03

The skewness, as measured by Sk., being not sensitive to deviations in the vertical direction, does not indicate clearly the degree of similarity between the sections. A direct comparison of corresponding ordinates of the sections was possible after

* The unit in which the standard deviations are measured corresponds in scale to the grouping unit of the observed variables.

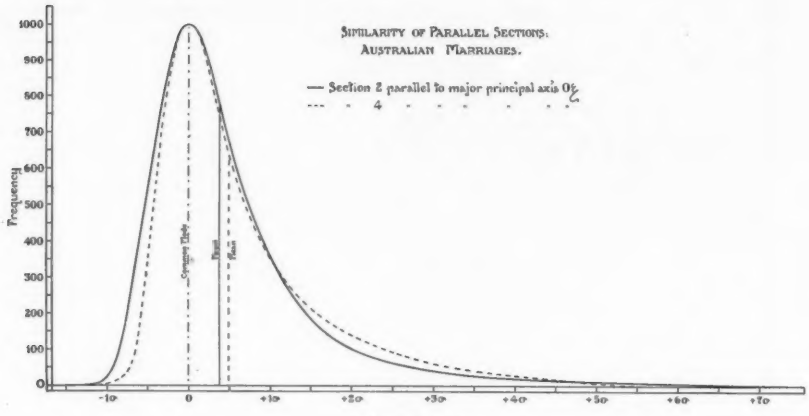

Diagram 1 (*i*).

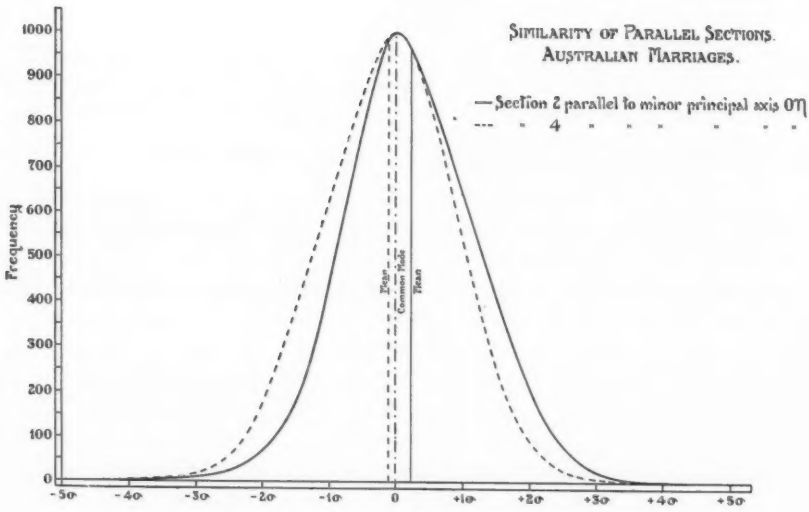

Diagram I (*j*).

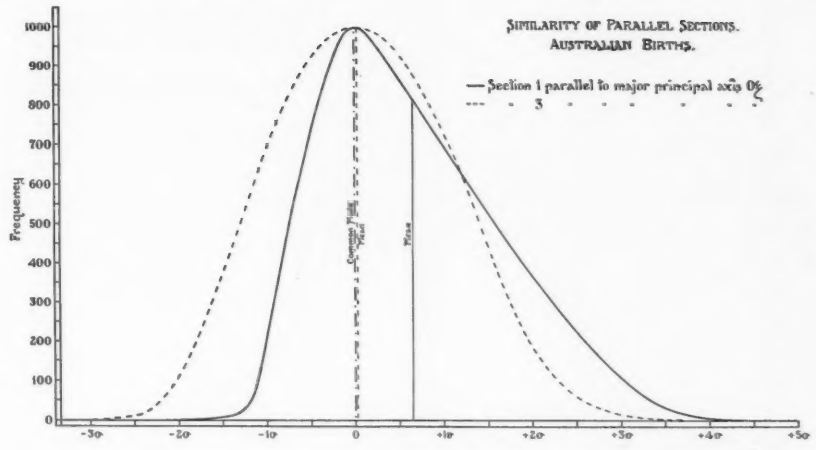

Diagram II (*i*).

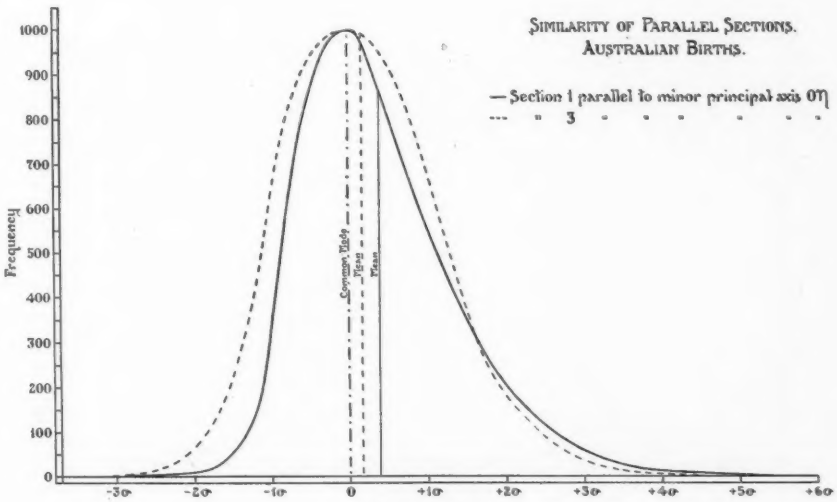

Diagram II (*j*).

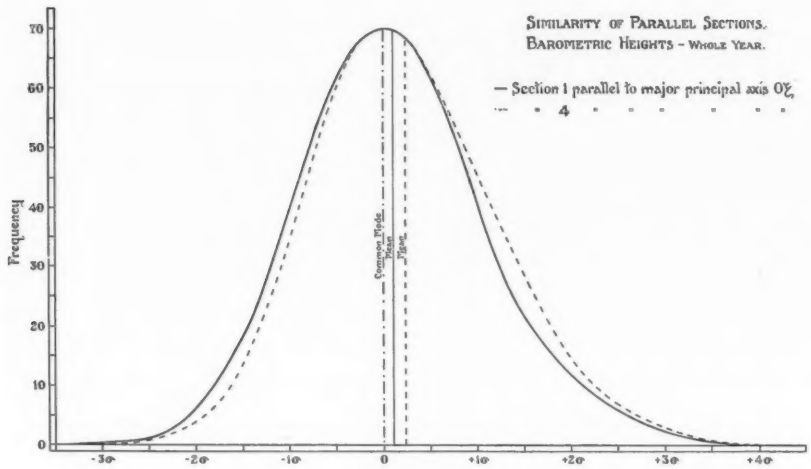

Diagram III (*i*).

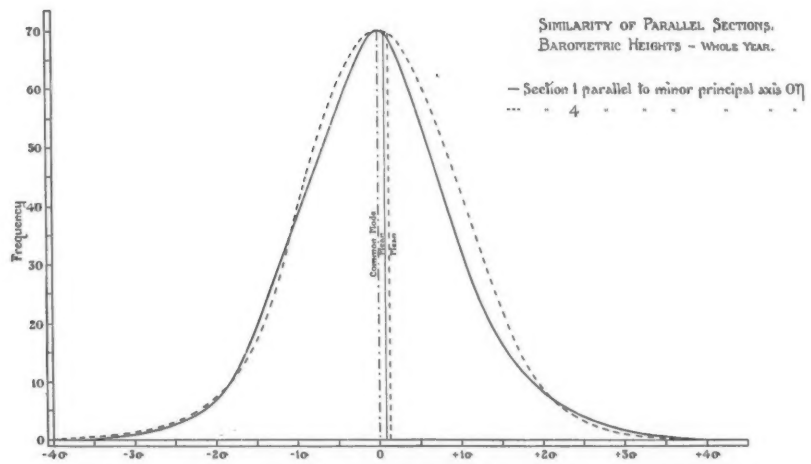

Diagram III (*j*).

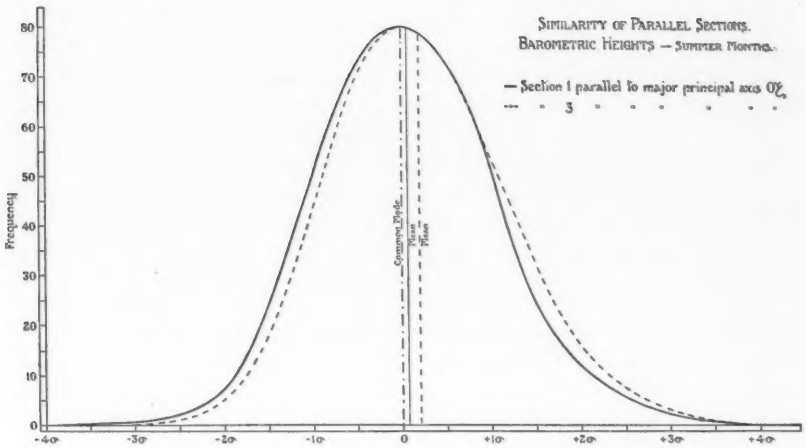

Diagram IV (*i*).

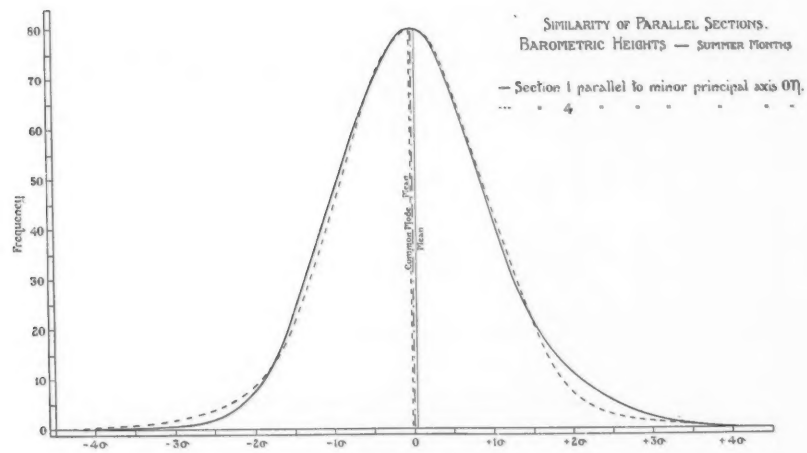

Diagram IV (*j*).

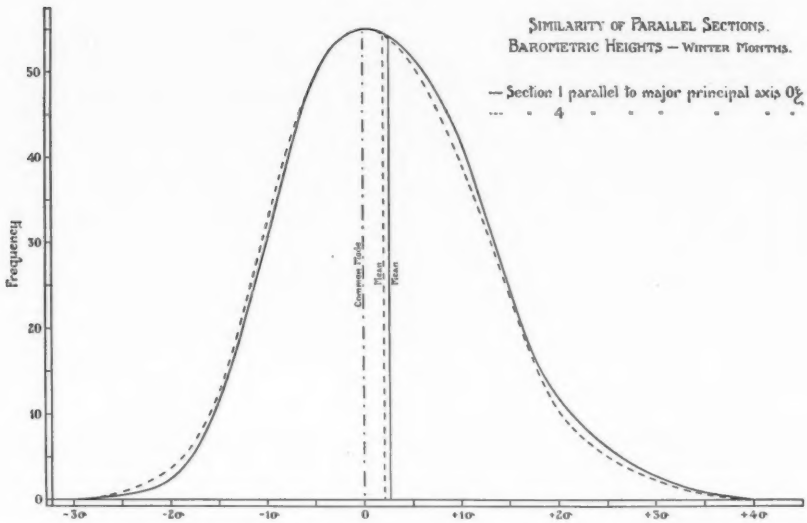

Diagram V (*i*).

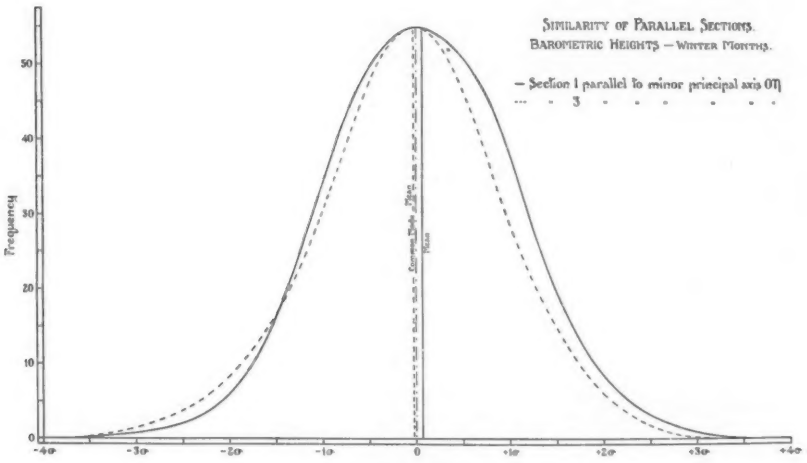

Diagram V (*j*).

both scales of measurement had been equalised, and on this the decision as to which two sections differ most was based. The sections parallel to $O\xi$ are, to all intents and purposes, homoscedastic; but this is the only instance where such a marked consistency was found. A rough appreciation of the heteroscedasticity of sections can be obtained from the range between the ordinates defined by, say, the lowest contour. Taking the sections parallel to $O\eta$, I find the range* of the most outlying contour (5) to have the following values:

Section 1, 12·37; Section 2, 14·04; Section 3, 9·43; Section 4, 6·12.

The ratio of the ranges of Sections 1 and 4, and of Sections 2 and 3 is ·495 and ·672 respectively; while the corresponding ratios of the standard deviations are ·511 and ·673. Such a close agreement can, of course, not always be expected; but for determining only approximately the "scedasticity," I think this method will be quite adequate.

The only contour systems about whose asymmetry there could have been some doubt are those of the barometric height distributions. The superposed sections, however, are of help in this issue. It is true that they do not definitely disprove the validity of an extended Narumi's hypothesis, but they show at least that a good fit cannot be expected in these cases from surfaces based on such an assumption. The hypothesis is not disproved because of the absence of a regular variation in the shape of the sections; it is not sanctioned because of the presence of rather significant irregularities.

4. *Identical Relations between the Moments.* The application of (*a*) Edgeworth's method of simple translation, (*b*) the Logarithmic surface, and (*c*) Rhodes' surface, depends on certain relations connecting the moments of the observed distributions being fulfilled. These criteria are contained in equations (29), (32), (68), and (54) respectively. The term "conditioned" will be used to designate the value of a constant computed from these equations.

(*a*) Equations (29) can be written in forms more convenient for treatment. On combining them, we find

$$\left.\begin{aligned}\sqrt{\beta_{10}} \equiv q_{30} &= \frac{3\,(2q_{21} - rq_{12})}{r\,(4 - r^2)}\\ \sqrt{\beta_{01}} \equiv q_{03} &= \frac{3\,(2q_{12} - rq_{21})}{r\,(4 - r^2)}\end{aligned}\right\} \quad \text{.....(70).}$$

If now, we assign to r, q_{21} and q_{12} in (70) their observed values, the "conditioned values" of β_{10} and β_{01} will be directly comparable with the observed as in the table on p. 206.

To bring out the significance of the difference between the two sets of values, I have tabulated in the last two rows the ratio

$$E = \frac{\text{Observed } \beta_1 - \text{"conditioned" } \beta_1}{\text{Standard Error† of "conditioned" } \beta_1}.$$

* Measured in the same unit as the standard deviations of the sections.

† Found from Table XXXVII in *Tables for Statisticians and Biometricians*, Part I.

The divergence testifies to Edgeworth's remark that simple translation is inadequate (p. 130).

		Table I	Table II	Table III	Table IV	Table V	Table VI
Observed	β_{10} β_{01}	4·058 3·854	·101 ·525	·203 ·204	·202 ·187	·142 ·143	·829 ·194
"Conditioned"	β_{10} β_{01}	6·502 4·958	·150 ·225	·084 ·128	·104 ·151	·035 ·059	1·072 ·314
E	β_{10} β_{01}	— —	32 71	12 6·2	6·5 2·1	23 12	2·3 3·6

(*b*) The determination of the constants of the Logarithmic Surface depends on moments and product moments up to the third order only. The relation between the fourth and lower moments is expressed in terms of the β's according to equations (68), whence the "conditioned" values of β_2 were calculated. For examining the fourth order product moments I assigned to the constants in equations (32) their observed values and then deduced the absolute product moments (q_{rs}) about the mean as origin. Only two distributions were considered, for it was clear from Diagram (4) that while the Logarithmic Normal Curve might hold for one of the marginal totals of some of the other distributions, it does not hold for both of them. The results summarised in the table below point to an approximate fulfilment of the relations between the moments. The effect, however, of this approximation on the shape of the surface had still to be investigated, and it was primarily with this end in view that I fitted the partial moment and marginal curves (pp. 208—218). The outcome, in short, is this: (i) that, if the third order moments and product moments are sufficient to give an account of the regression, they fail to do so for the scedasticity, clisy and kurtosis; (ii) that the relatively small deviation in β_{20} of the barometric data is sufficient to rank the Logarithmic Normal Curve as last, for goodness of fit, amongst four theoretical

		Table I	Table III			Table I	Table III
Observed	β_{20} β_{02}	9·290 8·333	3·398 3·399	Observed	q_{13} q_{22} q_{31}	6·297 6·346 7·070	1·865 1·817 1·881
"Conditioned"	β_{20} β_{02}	10·990 10·554	3·363 3·364				
E	β_{20} β_{02}	— —	·59 ·59	"Conditioned"	q_{13} q_{22} q_{31}	8·022 7·580 8·722	2·111 1·866 1·777

curves, the other three curves having all four moments equal to those observed; (iii) that notwithstanding the large deviations in the β_2's of the marriage data the Logarithmic Normal Curve gives as good a representation of the margins as the best-fitting Pearson curves. The equality of the fits in this case can be explained on the ground that when the β's are very high, the form of the frequency curve is not very sensitive to a change in them. Otherwise, we are bound to conclude that the agreement between the observed and "conditioned" values of the fourth order moments is not close enough for justifying the application of the logarithmic surface.

(*c*) The criteria justifying the use of Dr Rhodes' surface

$$\left.\begin{aligned} q_{21} - r\sqrt{\beta_{10}} &= \sqrt{\tfrac{1}{3}(1-r^2)(2\beta_{20}-3\beta_{10}-6)} \\ q_{12} - r\sqrt{\beta_{01}} &= \sqrt{\tfrac{1}{3}(1-r^2)(2\beta_{02}-3\beta_{01}-6)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(71)^*$$

were examined for all six distributions. In each case the "conditioned" β_2 was found by assuming for the other constants their observed values. In the table below E has the same meaning as in (*a*).

		Table I	Table II	Table III	Table IV	Table V	Table VI
Observed	β_{20}	9·290	2·430	3·398	3·274	2·862	4·863
	β_{02}	8·333	3·624	3·399	3·215	2·862	3·654
"Conditioned"	β_{20}	9·157	3·157	3·332	3·319	3·249	4·257
	β_{02}	8·803	3·960	3·325	3·288	3·243	3·388
E	β_{20}	—	80	1·2	·56	5	2·8
	β_{02}	—	14	1·3	·98	5	2·5

The conditions are satisfied best for the data of Tables I and IV; for Table I certainly far better than were the conditions of the Logarithmic Surface. Unfortunately, the partial moment and marginal curves, not being expressed in finite form, cannot be directly discussed. The disagreement obtained in the case of Table IV will be sufficient, I think, to affect appreciably the goodness of fit of the surface.

5. *The Marginal Distributions.* Pearson's system of curves, the Type A series, Edgeworth's translated curves, and the Logarithmic Normal Curve were used in specifying these distributions. Edgeworth's curves do not appear as marginal totals in his translated surfaces, but being at the basis of the method of translation as applied to correlation, they could not be left out of consideration.

* The equations show that, strictly, the surface cannot describe distributions whose marginal totals have β's lying below the Type III line, i.e. in the Type I area.

The probability integral of the Type A series is

$$\int_{-\infty}^{t} z_x dx = N\left[\tfrac{1}{2}(1 \pm a_t) - \frac{1}{\sqrt{6}}\sqrt{\beta_{10}}.\tau_3 - \frac{1}{\sqrt{24}}(\beta_{20}-3)\tau_4 - \frac{1}{\sqrt{120}}(\beta'_{30}-10\sqrt{\beta_{10}})\tau_5 - \frac{1}{\sqrt{720}}(\beta_4 - 15\beta_2 + 30)\tau_6 - \ldots\right]$$

$$\equiv N[\tfrac{1}{2}(1 \pm a_t) - a_3\tau_3 - a_4\tau_4 - a_5\tau_5 - a_6\tau_6 - \ldots] \quad \ldots\ldots\ldots\ldots\ldots\ldots(72),$$

where $t = x/\sigma_1$.

The equations of Edgeworth's curves and that of the Logarithmic Normal Curve are

$$\left.\begin{aligned} y &= \frac{N}{\sqrt{\pi}}.e^{-\xi^2} \\ x &= a(\xi + k\xi^2 + \lambda\xi^3) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(73),$$

and

$$y = \frac{N\log e}{\sqrt{2\pi}.s_1}.\frac{1}{x}.e^{-\frac{1}{2}\left(\frac{\log x - l_1}{s_1}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(74).$$

In referring to a particular frequency group, I shall state only its central value.

Marriage Statistics. The β's of the margins are

$$\beta_{10} = 4{\cdot}057{,}531, \quad \beta_{01} = 3{\cdot}853{,}680,$$
$$\beta_{20} = 9{\cdot}290{,}441, \quad \beta_{02} = 8{\cdot}332{,}812.$$

Turning to Diagram (1) we notice that both pairs of β's fall outside the area demarcated for the application of Edgeworth's curves. They suggest further the application of the Logarithmic Normal rather than of Type Aa. But before passing on to fitting that curve and the Pearson curves, we shall consider the correction of the moments for abruptness.

The sub-frequencies necessary for applying these corrections were obtained from the original table in Knibbs' *Mathematical Theory of Population*, the sub-intervals being single years. Discarding in the distribution of age of bride the observations in the lowest age-group, I find for the uncorrected and corrected moments about the beginning of the age-group 15·5 the following values:

Uncorrected	Sheppard's Corrections applied	Abruptness Corrections applied
$\nu_1' = 3{\cdot}907{,}280$	$\mu_1' = 3{\cdot}907{,}280$	$\mu_1' = 3{\cdot}907{,}140$
$\nu_2' = 20{\cdot}635{,}594$	$\mu_2' = 20{\cdot}282{,}261$	$\mu_2' = 20{\cdot}282{,}190$
$\nu_3' = 142{\cdot}046{,}089$	$\mu_3' = 141{\cdot}069{,}269$	$\mu_3' = 141{\cdot}069{,}626$

The moments of the distribution of age of bridegroom about the beginning of age-group 16·5 are:

Uncorrected	Sheppard's Corrections applied	Abruptness Corrections applied
$\nu_1' = 4{\cdot}794{,}355$	$\mu_1' = 4{\cdot}794{,}355$	$\mu_1' = 4{\cdot}794{,}353$
$\nu_2' = 30{\cdot}042{,}820$	$\mu_2' = 29{\cdot}959{,}486$	$\mu_2' = 29{\cdot}959{,}558$
$\nu_3' = 247{\cdot}854{,}858$	$\mu_3' = 246{\cdot}656{,}269$	$\mu_3' = 246{\cdot}656{,}423$

The slight difference between the values of the corresponding moments in the last two columns establishes for these cases the sufficiency of Sheppard's corrections.

(*a*) *x-Margin.* (i) The β's suggest a Pearson Type VI curve. The curve was fitted by fixing its start and by determining its constants from the first three moments. As a first trial the start was fixed at the beginning of the age-group 15·5; but this gave a too high frequency in the first group. The start was then fixed at a distance of 3·7 grouping units from the mean, i.e. at age = 14·622 years. This led to

$$y = y_0 (x - 5{\cdot}26753)^{4{\cdot}34930} . x^{-13{\cdot}96486},$$

where $\log y_0 = 15{\cdot}584{,}963$, the β's of the curve being

$$\beta_{10} = 4{\cdot}058{,}346, \quad \beta_{20} = 11{\cdot}706{,}379.$$

(ii) The constants of the Logarithmic Normal Curve found from equations (12) and (13) are

$$\xi_1 = 3{\cdot}73516, \quad l_1 = {\cdot}50562, \quad s_1 = {\cdot}24068.$$

The curve starts at 14·516 years, and its β's are

$$\beta_{10} = 4{\cdot}057{,}531, \quad \beta_{20} = 10{\cdot}990.$$

The frequencies are shown in Table IX. Notwithstanding a relatively large difference between the β_2's, the curves are markedly similar, there being not much to choose between them. Both curves tail off too slowly at the higher age-groups and fit badly at the lower; but here the material is undoubtedly spurious. Taking the largeness of the numbers into consideration, I think the graduation is not unsatisfactory.

(*b*) *y-Margin.* (i) Instead of the Type I_1 indicated by the β's, I tried a Type III and Type I_2 by fixing arbitrarily the start of the former curve, and the start, range or mode of the latter. But not one of the many trials that were made produced reasonable results; the curves could not be made to give correspondence at the tails as well as at the maximum frequencies. Finally a Type III

$$x = x_0 . e^{-\gamma y} (1 + y/a)^p$$

was fitted with start at the centre of the age-group 19·5. The position of the mode * was fixed by choosing $a = 1{\cdot}76$; while in assigning to p the value 1·2, I paid more attention to the modal frequency of the distribution than to its variability. The other constants are

$$\gamma = {\cdot}68182, \quad x_0 = 70004{\cdot}6,$$

the β's of the curve being $\beta_{01} = 1{\cdot}818$, $\beta_{02} = 5{\cdot}727$.

(ii) The Logarithmic Normal Curve fitted from the first three moments has

$$\eta_1 = 4{\cdot}49910, \quad l_2 = {\cdot}58884, \quad s_2 = {\cdot}23629, \quad \beta_{02} = 10{\cdot}554,$$

its start being at 15·886 years.

* A smoothing cubic fitted to the five largest frequencies about the mode gave the position of the mode at 1·8 grouping units from the centre of age-group 19·5.

TABLE IX.

Marginal Distributions of Ages of Brides and Bridegrooms, Australian Marriages.

	Brides			Bridegrooms			
Age of Bride	Observed Frequency	Theor. Freq. Type VI	Theor. Freq. Log. Normal	Age of Bridegroom	Observed Frequency	Theor. Freq. Type III	Theor. Freq Log. Normal
12·5	5	—	—	16·5	294	—	259
15·5	2,975	3,412	2,207	19·5	10,995	9,256	19,453
18·5	38,291	44,620	44,776	22·5	61,001	57,339	56,819
21·5	80,847	74,649	77,051	25·5	73,054	68,434	64,141
24·5	71,010	64,826	64,894	28·5	56,501	56,867	51,989
27·5	44,541	44,346	43,568	31·5	33,478	40,827	36,894
30·5	24,261	27,603	27,008	34·5	20,569	27,073	24,735
33·5	13,752	16,590	16,318	37·5	14,281	17,075	16,203
36·5	8,883	9,896	9,831	40·5	9,320	10,405	10,542
39·5	6,062	5,936	5,966	43·5	6,236	6,184	6,869
42·5	3,478	3,605	3,664	46·5	4,770	3,605	4,503
45·5	2,605	2,222	2,281	49·5	3,620	2,068	2,977
48·5	1,805	1,393	1,442	52·5	2,190	1,177	1,987
51·5	1,139	888	924	55·5	1,655	660	1,339
54·5	645	575	601	58·5	1,100	367	912
57·5	513	379	396	61·5	810	203	627
60·5	291	253	264	64·5	649	112	435
63·5	242	172	179	67·5	487	61	305
66·5	206	118	122	70·5	326	33	216
69·5	130	82	84	73·5	211	18	154
72·5	56	58	59	76·5	119	10	111
75·5	25	41	41	79·5	73		80
78·5	16	30	29	82·5	27	11	59
81·5	6	22	21	85·5	14		43
84·5	1	16	15	88·5	5	—	32
—	—	48	44	—	—	—	101
—	—			—	—	—	
Totals	301,785	301,780	301,785	—	301,785	301,785	301,785
χ^2	—	—	—	—	—	—	—
P	—	—	—	—	—	—	—

The frequencies are seen in Table IX. As could have been expected the Type III fails to give a description of the long tail; while the Logarithmic Normal, being better here than the Type III, is inferior to it at the mode. The divergence of the β's of the curves from those observed is noteworthy.

Birth Statistics. (*a*) *x-Margin.* The observed β's

$$\beta_{10} = \cdot 100{,}603, \quad \beta_2 = 2{\cdot}430{,}327$$

fall outside the compass of Edgeworth's method of translation; the distribution was used in Section C, 2 simply for illustrating the type of singularity to which the method of translation is subject.

(i) The equation of the Type I with start fixed at age 15·60*, and fitted from the first three moments, is

$$y = 75911{\cdot}6\left(1+\frac{x}{6{\cdot}12055}\right)^{1{\cdot}81499}\left(1-\frac{x}{12{\cdot}45834}\right)^{3{\cdot}69440}.$$

TABLE X.

Marginal Distribution of Ages of Mothers, Australian Births.

Age of Mother	Observed Frequency	Theor. Freq. Type I	Theor. Freq. Type Aa
—	—	—	} −1766
—	—	—	
13·0	3	—	391
15·0	191	46	3,570
17·0	4,573	6,105	10,482
19·0	21,322	22,871	22,049
21·0	42,758	41,996	37,545
23·0	62,620	58,455	54,745
25·0	73,423	69,796	67,472
27·0	74,834	75,176	75,848
29·0	72,640	74,822	76,574
31·0	65,182	69,637	71,313
33·0	58,407	60,909	62,121
35·0	48,834	50,071	50,833
37·0	39,932	38,524	38,804
39·0	31,050	27,485	27,222
41·0	18,975	17,878	17,237
43·0	11,283	10,359	9,678
45·0	4,365	5,088	4,734
47·0	1,072	1,943	1,973
49·0	199	476	676
51·0	13	44	175
53·0	4	1	} 6
55·0	2	—	
Totals	631,682	631,682	631,682
χ^2	—	—	—
P	—	—	—

(ii) For the Type Aa probability integral we have

$$a_3 = {\cdot}129{,}488, \quad a_4 = -{\cdot}116{,}284.$$

The theoretical frequencies tabulated in relation to the observed in Table X show a superiority of the Type I over the Type Aa; that curve, though, does not fit very well itself.

* Again many experiments were made to find the best-fitting curve.

(*b*) *y-Margin.* (i) The Pearson curve determined from the observed β's

$$\beta_{01} = \cdot 524{,}655, \quad \beta_{02} = 3 \cdot 624{,}169$$

is $$x = 105{,}928 \cdot 6 \left(1 + \frac{y}{4 \cdot 80170}\right)^{3 \cdot 64155} \left(1 - \frac{y}{43 \cdot 57116}\right)^{33 \cdot 04388}.$$

The start of the curve is at 16·089 years.

(ii) The Type A probability integral is

$$\int_{-\infty}^{t} z_y \, dy = N[\tfrac{1}{2}(1 \pm a_t) - \cdot 295{,}707\tau_3 - \cdot 127{,}408\tau_4 - \cdot 060{,}222\tau_5 - \cdot 234{,}704\tau_6 - \ldots],$$

where $t = y/\sigma_2$. The higher β's were found from the observations; the series was fitted up to τ_4 only (≡ Type Aa).

(iii) The parameters in Edgeworth's curve have the following values:

$$\lambda = -\cdot 00717, \quad k = \cdot 17717, \quad a = 3 \cdot 51098.$$

The median is at 32·567 years.

(iv) For the Logarithmic Normal Curve

$$\eta_1 = 10 \cdot 52763, \quad l_2 = 1 \cdot 01046, \quad s_2 = \cdot 10153.$$

The start of the curve is at 1·917 years; the "conditioned" $\beta_{02} = 3 \cdot 947$.

The corresponding frequencies are compared in Table XI. A "break" occurs in Edgeworth's curve at age 19·1, corresponding to a deviation of $\xi = -2 \cdot 456$* from the mean of the normal curve. About one-thousandth part of the area under the normal curve is thus folded over. Beyond this "break" the curve fits very well but loses its advantage of course if the first two groups be taken into consideration; the Type I is then to be preferred. In the tetrachoric series the τ_5 and τ_6 terms are evidently not negligible; while the large difference between the observed β_{02} and that of the Logarithmic Normal Curve accounts for the failure of this curve.

Barometric Heights (*Whole Year*). The two marginal columns being practically similar, I examined only one of them: the *x-margin*. The observed β's

$$\beta_{10} = \cdot 203{,}214, \quad \beta_{20} = 3 \cdot 397{,}763$$

lie in the Type IV area very close to the Type V† line; they conform also approximately to the relation between the β's of the Logarithmic Normal Curve.

The theoretical curves with their constants are as follows:

(i) Pearson's Type IV:

$$y = y_0 \left(1 + \frac{x^2}{91 \cdot 56833}\right)^{-36 \cdot 41804} \times e^{176 \cdot 85828 \tan^{-1} \frac{x}{9 \cdot 56913}},$$

where $$\log y_0 = \overline{57} \cdot 481{,}845.$$

(ii) Type Aa probability integral:

$$a_3 = \cdot 184{,}036, \quad a_4 = \cdot 081{,}193.$$

(iii) Edgeworth's translated curve:

$$\lambda = \cdot 01045, \quad k = \cdot 10559, \quad a = 4 \cdot 26484,$$

the median being at 29·803″. The curve presents no singularities in this case.

* There is another break at $\xi = +18 \cdot 939$.

† The Type V, with the observed β_{10}, has its $\beta_{20} = 3 \cdot 385$ and gives $\chi^2 = 72 \cdot 39$.

TABLE XI.

Marginal Distribution of Ages of Fathers, Australian Births.

Age of Father	Observed Frequency	Theor. Freq. Type I	Theor. Freq. Type Aa	Theor. Freq. Edgeworth	Theor. Freq. Log. Normal
—	—	—	{ −705	—	{ 82
—	—	—		—	
16·5	181	{ 131	3,154	{ ?	1,689
19·5	7,936	{ 9,175	13,464	{ 8,918	11,342
22·5	40,789	39,776	35,168	41,244	36,613
25·5	79,964	76,959	66,936	76,912	71,852
28·5	99,328	100,662	97,806	99,682	99,451
31·5	102,303	104,277	112,097	103,614	107,052
34·5	90,670	92,260	102,082	92,246	95,787
37·5	73,609	72,643	75,335	72,981	74,643
40·5	52,930	52,183	47,500	52,510	52,376
43·5	35,507	34,744	28,799	34,904	33,913
46·5	21,817	21,669	19,101	21,675	20,639
49·5	12,781	12,748	13,527	12,682	11,968
52·5	6,717	7,109	8,920	7,035	6,684
55·5	3,587	3,775	4,991	3,719	3,626
58·5	1,821	1,908	2,294	1,882	1,922
61·5	911	921	862	914	1,001
64·5	489	425	266	427	515
67·5	183	187	68	193	262
70·5	85	79	14	84	132
73·5	38	32	{ 3	35	67
76·5	25	{ 12		{ 15	33
79·5	9	{ 4	—	{ 6	17
82·5	2	{ 3	—	{ 4	8
—	—	—	—	—	{ 8
—	—	—	—	—	
Totals	631,682	631,682	631,682	631,682	631,682
χ^2	—	480·4	—	299·4	—
P	—	·000	—	·000	—

(iv) Logarithmic Normal Curve:

$$\xi_1 = 20{\cdot}64737, \quad l_1 = 1{\cdot}31009, \quad s_1 = {\cdot}06442.$$

The curve starts at 31·846″ and its "conditioned" $\beta_{20} = 3{\cdot}363$.

The frequencies are seen in Table XII. It is obvious that the first three curves, with both their β's equal to those of the data, are superior to the Logarithmic Normal Curve—the difference between the observed and computed β_2 being only about ·6 of the standard error of the latter. The values of χ^2 are not directly comparable; the number of categories is less for the Type Aa than for the other representations. If this be borne in mind, then there is not much to choose between the first three sets of results.

TABLE XII.

x-Marginal Distribution of Barometric Heights (Whole Year).

Barometric Height	Observed Frequency	Theor. Freq. Type IV	Theor. Freq. Type Aa	Theor. Freq. Edgeworth	Theor. Freq. Log. Normal
—	—				
30·75	7	2	·5	2·6	1·4
30·65	13	13	19·6	14·0	11·0
30·55	73	63	81·3	63·7	58·6
30·45	258	220	243·9	218·7	215·7
30·35	563	583	583·6	579·1	584·8
30·25	1148	1219	1165·8	1214·1	1229·5
30·15	1951	2074	1976·6	2071·6	2087·2
30·05	2951	2948	2878·0	2952·8	2957·3
29·95	3749	3595	3623·7	3601·7	3592·0
29·85	3921	3835	3966·9	3840·8	3823·8
29·75	3700	3648	3798·1	3648·3	3633·8
29·65	3176	3140	3209·8	3136·7	3130·4
29·55	2333	2479	2434·0	2475·2	2476·6
29·45	1752	1817	1703·3	1813·4	1819·1
29·35	1233	1247	1143·8	1246·0	1252·2
29·25	813	810	763·9	809·6	814·4
29·15	542	501	511·2	501·2	503·8
29·05	282	296	333·9	297·4	298·3
28·95	189	169	204·9	170·0	169·9
28·85	81	93	114·6	94·1	93·5
28·75	60	50	57·3	50·6	49·9
28·65	43	27	25·4	26·5	25·9
28·55	12	13	9·8	13·6	13·2
28·45	4	7	3·6	6·8	6·6
28·35	1	6	1·5	3·4	3·2
—	—			3·1	2·9
—	—	—	—		
Totals	28,855	28,855	28,855	28,855	28,855
χ^2	—	68·54	61·95	64·96	79·63
P	—	·000	·000	·000	·000

Barometric Heights. x-Margins of (a) Summer and (b) Winter Months.*

(*a*), (i) The β's satisfy approximately the condition of the Type III curve. The resulting equation is

$$y = 2465{\cdot}72\,.\,e^{-1{\cdot}84384x}\left(1+\frac{x}{10{\cdot}21891}\right)^{18{\cdot}84204}.$$

The start of the curve is at 30·866″.

(ii) The coefficients of the tetrachoric terms are

$$a_3 = {\cdot}183{,}299, \quad a_4 = {\cdot}056{,}014.$$

* Edgeworth's curve was not fitted to these distributions, for they are of more or less the same degree of skewness as the distribution considered in the last paragraph.

(*b*), (i) The appropriate Pearson curve is

$$y = 1519{\cdot}38\left(1+\frac{x}{8{\cdot}86517}\right)^{3{\cdot}62656}\left(1-\frac{x}{22{\cdot}08814}\right)^{9{\cdot}03580}$$

with its start at 30·748″.

(ii) The Type Aa integral has

$$a_3 = {\cdot}153{,}949, \quad a_4 = -{\cdot}028{,}267.$$

The frequencies for both margins are shown in Table XIII. The goodness of fit, as measured by P, manifests the superiority of the Pearson curves over the Type Aa.

TABLE XIII.

x-Marginal Distributions of Barometric Heights (Summer and Winter Months).

	Summer Months			Winter Months		
Barometric Height in inches	Observed Frequency	Theor. Freq. Type III	Theor. Freq. Type Aa	Observed Frequency	Theor. Freq. Type I	Theor. Freq. Type Aa
—	—	—	—	—	—	{
30·75	—	—	—	7	{	{ −11·1
30·65	—	—	—	13	11·6	{ 48·7
30·55	—	{	{	73	81·3	129·7
30·45	10	10·5	11·20	248	250·8	270·2
30·35	81	79·0	102·6	482	508·9	478·3
30·25	338	329·9	344·3	810	811·6	743·9
30·15	830	862·2	825·4	1121	1103·0	1034·1
30·05	1576	1580·3	1513·1	1375	1334·8	1298·8
29·95	2256	2196·3	2182·3	1493	1475·6	1485·7
29·85	2427	2446·4	2516·2	1494	1514·7	1559·0
29·75	2279	2274·1	2353·8	1421	1458·7	1511·7
29·65	1812	1818·4	1827·2	1364	1327·0	1365·8
29·55	1236	1280·4	1226·1	1097	1145·1	1159·7
29·45	822	808·7	756·4	930	939·6	933·6
29·35	484	464·8	452·9	749	733·8	717·7
29·25	252	246·1	262·7	561	545·2	528·8
29·15	122	121·2	140·0	420	384·7	373·2
29·05	47	56·0	64·8	235	257·2	251·2
28·95	27	24·4	25·2	162	162·1	160·1
28·85	9	{ 10·1	{ 8·1	72	95·8	96·0
28·75	5	{ 4·0	{ 2·1	55	52·7	53·7
28·65	2	{ 2·2	{ ·6	41	26·6	27·9
28·55	—	{	{	12	{ 12·2	13·5
28·45	—	—	—	4	{ 4·9	{ 6·0
28·35	—	—	—	1	{ 2·1	{ 3·8
—	—	—	—	—	{	{
Totals	14,615	14,615	14,615	14,240	14,240	14,240
χ^2	—	7·73	33·76	—	34·49	89·64
P	—	·956	·006	—	·032	·000

Length and Breadth of Beans. (*a*) *x-Margin.*

(i) The observed β's

$$\beta_{10} = \cdot 829{,}136, \quad \beta_{20} = 4 \cdot 862{,}944$$

led to the following Pearson Type IV curve

$$y = \cdot 395{,}121 \left(1 + \frac{x^2}{17 \cdot 30134}\right)^{-8 \cdot 34886} \cdot e^{-18 \cdot 38043 \tan^{-1} \frac{x}{4 \cdot 15949}}.$$

(ii) An extensive trial with the Type A series was made in this case; successive approximations, including the Type Ab (see p. 114), were considered. These are denoted in Table XIV by the symbol Σ, the suffix indicating the order of the last tetrachoric term in the approximation. The higher β's involved in the coefficients of the tetrachoric terms have the observed values

$$\beta'_{30} = -12 \cdot 574{,}125, \quad \beta_{40} = 53 \cdot 221{,}083.$$

TABLE XIV.

Marginal Distribution of Length of Beans.

Length of Beans in mm.	Observed Frequency	Theor. Freq. Type IV	Theor. Freq. Type Aa $\equiv \Sigma_4$	Theor. Freq. Type A $\equiv \Sigma_5$	Theor. Freq. Type A $\equiv \Sigma_6$	Theor. Freq. Type Ab $\equiv \Sigma_{4\cdot6}$	Theor. Freq. Edgeworth	Theor. Fre Log. Norm
—	—	—	16·3	−15·2	2·0	30·2	—	—
—	—	1·4	12·8	13·7	−35·3	−26·8	4·8	10·1
17·0	6	28·5	25·6	116·6	22·3	−50·6	38·6	280·5
16·5	55	299·3	241·7	370·4	438·1	293·4	280·4	1255·2
16·0	275	1181·6	1012·7	926·2	·1214·0	1244·9	1136·6	2163·8
15·5	1129	2132·6	2155·4	1833·0	1866·9	2182·7	2177·9	2179·6
15·0	2082	2229·8	2593·0	2506·4	2112·8	2275·6	2269·5	1598·5
14·5	2294	1638·9	1788·4	2082·6	1916·7	1654·6	1625·3	965·0
14·0	1787	968·9	713·4	921·3	1183·4	924·8	948·2	515·3
13·5	929	503·6	280·7	199·0	371·2	419·6	495·6	254·5
13·0	437	243·7	258·7	132·1	66·9	206·1	244·2	119·5
12·5	199	113·8	206·2	178·1	101·2	144·2	116·5	54·4
12·0	115	52·5	98·7	117·0	107·1	90·7	54·7	24·3
11·5	70	24·2	29·6	43·5	54·0	38·2	25·5	10·7
11·0	36	11·3	5·9	10·0	15·4	10·3	11·8	4·8
10·5	18	5·4	·9	1·7	3·3	2·1	5·5	2·1
10·0	7	2·6	—	—	—	—	2·6	1·7
9·5	1	1·9					2·3	
—	—	—					—	—
—	—							
Totals	9440	9440	9440	9440	9440	9440	9440	9440
χ^2	—	50·3	325·9	608·9	425·3	85·6	46·04	89·0
P	—	·000	—	—	—	—	·000	—

Whence we get

$$\int_{-\infty}^{t} z_x \, dx = N[\tfrac{1}{2}(1 \pm a_t) + \cdot 371{,}738\tau_3 - \cdot 380{,}272\tau_4 + \cdot 316{,}623\tau_5 - \cdot 382{,}998\tau_6 + \ldots].$$

The coefficient of $\tau_6 = -\cdot 309{,}001$ in the Type Ab approximation.

(iii) Edgeworth's curve has

$$\lambda = \cdot 05836, \quad k = \cdot 20676, \quad a = 2 \cdot 29416;$$

its median is at 14·523 mm.

(iv) The Logarithmic Normal has

$$\xi_1 = 6 \cdot 10087, \quad l_1 = \cdot 76728, \quad s_1 = \cdot 12544;$$

it starts at 17·445 mm. and its $\beta_{20} = 4 \cdot 509$.

The frequencies are compared in Table XIV. The practical non-convergency of the Type A expansion, which is suggested by the coefficients of the successive terms in the series, is brought out clearly by the results in columns 4, 5 and 6. The effect of the τ_5 and τ_6 terms is to reduce instead of to increase the goodness of fit. Moreover, there is a hump in the curves at x about 12 mm. which is contrary to the nature of the data. The Type Ab produces negative frequencies at the high values of the variable, but beyond this it fits better than any one of the other approximations. I fail, however, to find any *a priori* justification, in this case, for neglecting the τ_5 term. Edgeworth's curve gives the best representation of the data; it presents no singularities and fits better than the Type IV at the high values of the variable. The Logarithmic Normal is rather unsatisfactory, especially at its start.

(*b*) *y-Margin.* The observed β's

$$\beta_{01} = \cdot 194{,}333, \quad \beta_{02} = 3 \cdot 654{,}374$$

led to the following curves:

(i) Pearson's Type IV

$$x = 266 \cdot 445 \left(1 + \frac{y^2}{28 \cdot 19141}\right)^{-11 \cdot 16896} \cdot e^{-10 \cdot 52409 \tan^{-1} \frac{y}{5 \cdot 30956}}$$

(ii) Type Aa integral

$$a_3 = -\cdot 179{,}969, \quad a_4 = \cdot 133{,}574.$$

(iii) Edgeworth's curve

$$\lambda = \cdot 03079, \quad k = \cdot 10055, \quad a = 1 \cdot 82812,$$

with its median at 7·998 mm. Again there are no singularities.

The frequencies are given in Table XV. The goodness of fit, as measured by P, shows a slight superiority of Edgeworth's curve over the Type IV, and a distinct superiority of these two curves over the Type Aa.

The results of this section point to the following main conclusions:

(i) Of the curves we dealt with, the Logarithmic Normal is easiest to apply; but because of the relation between its fourth and lower moments it is of small practical use. When β_1 and β_2 are not large, even a relatively small deviation from this relation affects appreciably the goodness of fit of the curve.

TABLE XV.

Marginal Distribution of Breadth of Beans.

Breadth of Beans	Observed Frequency	Theor. Freq. Type IV	Theor. Freq. Type Aa	Theor. Freq. Edgeworth
—	—	—	1·4	—
—	—	3·0	7·0	4·3
9·125	5	48·6	50·4	49·4
8·875	48	393·6	374·8	385·7
8·625	400	1515·4	1474·6	1513·6
8·375	1483	2732·5	2793·3	2749·7
8·125	2742	2530·5	2609·8	2528·2
7·875	2579	1413·3	1321·8	1403·0
7·625	1397	557·9	516·3	557·4
7·375	530	177·2	213·5	179·7
7·125	170	49·9	65·0	51·1
6·875	72	13·3	11·0	13·4
6·625	10	4·8	1·1	4·5
6·375	4			
—	—			
Totals	9440	9440	9440	9440
χ^2	—	14·38	18·18	13·50
P	—	·110	·033	·140

(ii) As a general frequency function the Type A expansion* with either of the approximations to it, the Type Aa or Type Ab, is inferior to the Pearson or Edgeworth curves. It is simple to apply, but unless the distributions be only moderately skew, this simplicity is not commensurate with the accuracy sacrificed.

(iii) Edgeworth's translated curves, although they also have definite limits of applicability, have a much wider range than the Type A series has. As set out in Section C, 2 the method is not a universal one for describing frequency distributions; it is discredited for all but a small portion of the Pearson Type I area. Within the set limits, however, they give as good a representation of the data as the Pearson curves do. The method has the disadvantage that the start of the curves cannot be fixed at will; and if a high degree of accuracy is desired, the determination of the fundamental constants as well as the solution of a number of cubic equations becomes rather lengthy; it has the advantage that the theoretical frequencies are directly obtainable.

(iv) The Pearson curves are the most useful and most general of those considered; they will describe almost every variety of distribution we have to deal with in practice; the start of the non-symmetrical curves, except in the case of

* See also the remarks on pp. 113—114.

Type IV, is adjustable to the nature of the data. The greatest drawback of the system lies at present in the amount of numerical work it entails.

F. *General Considerations.*

In Section C, I gave an account of the contributions that had been made to the solution of the problem of describing mathematically skew bivariate frequency distributions. After having further analysed, in Section D, some of the characteristics of two or three of the proposed frequency functions, I passed on to a detailed graphical analysis of six observed distributions with the object of studying the generality of the theoretical surfaces. These results will now be embodied in a short discussion of the more outstanding aspects of the problem.

As a general starting-point from which a system of skew surfaces can be developed, the idea of axes of independent probability does not appear of great value; the heteroscedasticity of sections parallel to the principal axes through the modes of the distributions, such as we have actually found to exist, proves the interdependency of the transformed variables. Of course, this method might be adequate in certain isolated cases, but as its application will generally have to be justified by a comparison with other representations, there seems to be no point in using it.

The method of linear transformation suggested by Steffensen includes the above as a special case. It, too, consists in replacing the correlation function by the simple product of two univariate functions, but the axes representing these new variables are not necessarily rectangular; the angle they enclose is determined from the moments about the original axes. It is under this group, I think, that Dr Rhodes' surface may be classed. Although these formulae have the advantage that they can be applied with relative ease, their breadth of application is limited by their not allowing for enough freedom in the variation of the array and marginal distributions. The small number of parameters they contain requires certain relations between at least the fourth and lower order momental constants to be satisfied. But such relations seem rarely to exist in practice; those pertaining to the Rhodes' surface are satisfied approximately by only one of the distributions considered. Apart from these limitations, the formulae suffer from the serious disadvantage that their array moment and marginal curves are not finite expressions and thus not of any direct practical use.

The supposition of homoclitic and homokurtic array-sections from which Narumi developed his system of surfaces is not supported by our numerical illustrations. The data of Tables I—V involve (entangled with an irregular variation) a regular variation in the shape of the arrays; for the arrays of Table VI no regularity, whether changing or constant, is discernible. I am inclined to think, in fact, that the conditions presupposed by Narumi will physically very seldom be realised if each variable has a more or less distinct upper limit and if neither of these limits is a function of the other variable concerned. Thus, if there be positive correlation the x (or, y)-variable attains a greater freedom of variation in the direction of its upper limit as y (or, x) decreases in value. The arrays tend to become more and more symmetrical and may ultimately even become skew in the

opposite direction if the variables have lower limits as well. This is what actually happens in the distributions with which we have been dealing. While the condition of similar parallel sections seems likely to be fulfilled sooner by the transformed variables represented by the principal axes through the mode or mean of the distribution, than by the observed variables, the enquiry we made as to this led to rather discouraging results; the method is moreover handicapped by the unwieldiness of the resulting array moment curves.

The direct extension of a system of frequency curves to the corresponding system of surfaces is, perhaps, the most natural way of approaching the skew-correlation problem. Several such attempts have been made involving the Pearson curves, Edgeworth's translated curves, the Type A series, and the logarithmically transformed Normal curve. Accordingly it was necessary to make a comparative study of these curves; the conclusions arrived at place the curves in the above order for general adequacy in representing observed data. Owing to the difficulty of mathematical analysis, it has up to the present not been possible to derive a general system of surfaces from the Pearson curves. Edgeworth extended his method of translation to two dimensions and discussed the two cases where the equations of translation involve the observed variables (i) separately: simple translation; and (ii) conjointly: composite translation. The relations between the moments which justify simple instead of composite translation are not even approximately fulfilled by our distributions; while composite translation is not feasible. It is further to be observed that here, too, the array moment curves and marginal curves cannot be directly discussed. In summarising the relative advantages of the Pearson curves and the method of translation, Edgeworth claimed as a proposition in favour of his system that "it is adapted to the representation of frequency *surfaces*"; the proposition is true, but in the light of the analysis we have made it carries hardly any weight.

The Type AaAa (or, 15-Constant) surface still remains the most general of the surfaces that have been propounded. Its value lies more in its having 15 available constants, however, than in the form of its equation. Its limitations are essentially the same as those of the Type A expansion which we discussed in Sections C, 2 and E, 5. The simplified form of the Type AA surface suggested by Jørgensen, where the generating surface is the simple product of two normal curves, is still less efficient as a skew bivariate frequency function. The differential terms alone are inadequate to give an account of the correlation. The Logarithmic surface, equation (30), is fully specified as soon as all the moments up to the third order are known; it has consequently only 10 available constants; the relations between its fourth and lower order moments severely restrict the generality of its application. For a discussion of the regression, scedastic, clitic, and kurtic curves of these two surfaces, we refer the reader to Section E, 2. From the scedastic curves onwards the fits obtained are rather unsatisfactory; but having only one surface with 15 constants and one with 10 at our disposal, we do not know what goodness of fit really can be obtained in fitting theoretical array moment curves to observational data. A consideration of other characteristics of the surfaces, however, leads

one to anticipate that better results could be obtained only with surfaces whose sections and marginal totals are more successful than the Type A and the Logarithmic Normal Curve have been found to be in representing observed univariate distributions.

It can still be said, in conclusion, that after more than thirty years the problem still remains the "most urgent task before mathematical statisticians." The solutions that have been put forward will be serviceable in certain special cases; but no satisfactory solution to the general problem has yet been reached.

I wish to acknowledge with gratitude my indebtedness to Professor Karl Pearson for proposing the subject of this study and for his continual kindly advice and criticism. I am also indebted to Miss Ida McLearn and Miss Joyce Townend for preparing the diagrams so well.

BIBLIOGRAPHY*.

ÅKESSON, O. A.: "On the Dissection of Correlation Surfaces." *Ark. f. Mat., Astr. och Fysik*, Bd. XI, No. 16, 1916.

BOWLEY, A. L.: "F. Y. Edgeworth's Contributions to Mathematical Statistics." *Roy. Stat. Soc.*, 1928.

CAMP, BURTON H.: "Mutually Consistent Multiple Regression Surfaces." *Biometrika*, Vol. XVII, 1925, pp. 443—458.

CHARLIER, C. V. L.: "Über das Fehlergesetz." *Ark. f. Mat., Astr. och Fysik*, Bd. II, No. 8, 1905.

—— "Die Zweite Form des Fehlergesetzes." *Ark. f. Mat., Astr. och Fysik*, Bd. II, No. 15, 1905.

—— "Über die Darstellung willkürlicher Funktionen." *Ark. f. Mat., Astr. och Fysik*, Bd. II, No. 20, 1905.

—— "Weiteres über das Fehlergesetz." *Ark. f. Mat., Astr. och Fysik*, Bd. IV, No. 13, 1907.

—— "Die strenge Form des Bernoulli'schen Theorems." *Ark. f. Mat., Astr. och Fysik*, Bd. V, No. 15, 1909.

—— "Contributions to the Mathematical Theory of Statistics. 5. Frequency Curves of Type A in Heterograde Statistics." *Ark. f. Mat., Astr. och Fysik*, Bd. IX, No. 25, 1914.

—— "Contributions to the Mathematical Theory of Statistics. 6. The Correlation Function of Type A." *Ark. f. Mat., Astr. och Fysik*, Bd. IX, No. 26, 1914.

CHARLIER, C. V. L. and WICKSELL, S. D.: "On the Dissection of Frequency Functions." *Ark. f. Mat., Astr. och Fysik*, Bd. XVIII, No. 6, 1923.

DAVIES, G. R.: "The Logarithmic Curve of Distribution." *Journ. Am. Stat. Ass.*, Dec. 1925, pp. 467—480.

—— "The Analysis of Frequency Distributions." *Journ. Am. Stat. Ass.*, Dec. 1929, pp. 349—366.

EDGEWORTH, F. Y.: "The Compound Law of Error." *Phil. Mag.*, Vol. XLI, 1896, pp. 207—215.

—— "The Law of Error." *Camb. Phil. Trans.*, Vol. XX, 1905, pp. 36—65, 113—141.

—— "On the Use of Analytical Geometry to Represent Certain Kinds of Statistics." *Journ. Roy. Stat. Soc.*, Vol. LXXVII, 1914, pp. 838—852.

—— "On the Mathematical Representation of Statistical Data." *Journ. Roy. Stat. Soc.*, Vol. LXXX, 1917, pp. 266—288.

GALTON, FRANCIS: "The Geometric Mean, in Vital and Social Statistics." *Roy. Soc. Proc.*, Vol. XXIX, 1879, pp. 365—367.

—— "Family Likeness in Stature." With an Appendix by J. D. Hamilton Dickson. *Roy. Soc. Proc.*, Vol. XL, 1886, pp. 42—73.

* This bibliography does not profess to be complete. I have indeed consulted a number of papers not here included, selecting those which seemed most important for my present purposes.

HENDERSON, JAMES: "On Expansions in Tetrachoric Functions." *Biometrika*, Vol. XIV, 1922–23, pp. 157—185.

ISSERLIS, L.: "The Application of Solid Hypergeometrical Series to Frequency Distributions in Space." *Phil. Mag.*, Vol. XXVIII, 1914, pp. 379—403.

JØRGENSEN, N. R.: "Note sur la fonction de répartition de Type B de M. Charlier." *Ark. f. Mat., Astr. och Fysik*, Bd. X, No. 15, 1914.

—— "Undersøgelser over Frequensflader og Korrelation." Arnold Busck, København, 1916.

KAPTEYN, J. C. and VAN UVEN, M. J.: "Skew Frequency Curves in Biology and Statistics." Hoitsema Bros., Groningen, 1916.

MACALISTER, D.: "The Law of the Geometric Mean." *Roy. Soc. Proc.*, Vol. XXIX, 1879, pp. 367—376.

NARUMI, SEIMATSU: "On the General Forms of Bivariate Frequency Distributions which are Mathematically Possible when Regression and Variation are subjected to Limiting Conditions." *Biometrika*, Vol. XV, 1923, pp. 77—88, 209—221.

NEYMAN, J.: "Further notes on Non-Linear Regression." *Biometrika*, Vol. XVIII, 1926, pp. 257—262.

PEARSON, KARL: "On the General Theory of Skew Correlation and Non-linear Regression." *Drapers' Company Research Memoirs*, Biometric Series II, 1905.

—— "'Das Fehlergesetz und seine Verallgemeinerungen durch Fechner und Pearson.' A Rejoinder." *Biometrika*, Vol. IV, 1905–6, pp. 169—212.

—— "Notes on the History of Correlation." *Biometrika*, Vol. XIII, 1920, pp. 25—45.

—— "On a General Method of Determining the Successive Terms in a Skew Regression Line." *Biometrika*, Vol. XIII, 1921, pp. 296—300.

—— "Notes on Skew Frequency Surfaces." *Biometrika*, Vol. XV, 1923, pp. 222—230.

—— "Non-Skew Frequency Surfaces." *Biometrika*, Vol. XV, 1923, pp. 231—244.

—— "On a Certain Double Hypergeometrical Series and its Representation by Continuous Frequency Surfaces." *Biometrika*, Vol. XVI, 1924, pp. 172—204.

—— "The Fifteen Constant Bivariate Frequency Surface." *Biometrika*, Vol. XVII, 1925, pp. 268—313.

—— "The Contribution of Giovanni Plana to the Normal Bivariate Frequency Surface." *Biometrika*, Vol. XX A, 1928, pp. 295—298.

PEROZZO, LUIGI: "Nuove Applicazioni del Calcolo delle Probabilità." *Reale Accademia dei Lincei*, Serie 3ª, *Memorie della Classe di scienze morali, storiche e filologiche*, Vol. X, 1881–2, pp. 1—33.

RHODES, E. C.: "On a Certain Skew Correlation Surface." *Biometrika*, Vol. XIV, 1922–3, pp. 355—377.

—— "On a Skew Correlation Surface." *Biometrika*, Vol. XVII, 1925, pp. 314—326.

SCHOLS, C. H. M.: "Théorie des Erreurs dans le Plan et dans l'Espace." *Ann. de l'École Polytechn. de Delft*, 1886, pp. 123—175. Published in Dutch in the *Verh. v. Kon. Ak. v. Wet.*, D. XV, 1875, Amsterdam.

STEFFENSEN, J. F.: "A Correlation Formula." *Sk. Akt. Tidskr.*, 1922.

—— "Matematisk Iagttagelselære." G. E. C. Gads, København, 1923.

—— "On Charlier's Generalized Frequency Function." *Sk. Akt. Tidskr.*, 1924.

VAN UVEN, M. J.: "On Treating Skew Correlation." *Proc. Kon. Ak. v. Wet.*, Amsterdam, Vol. XXVIII, Nos. 8—9, 1925, pp. 797—811; Vol. XXVIII, No. 10, 1925, pp. 919—935; Vol. XXIX, No. 4, 1926, pp. 580—590; Vol. XXXII, No. 4, 1929, pp. 408—413.

—— "Skew Correlation between Three and More Variables." *Proc. Kon. Ak. v. Wet.*, Vol. XXXII, No. 6, 1929, pp. 793—807; No. 7, pp. 995—1007; No. 8, pp. 1085—1103.

WALKER, HELEN M.: "The Relation of Plana and Bravais to the Theory of Correlation." *Isis*, Vol. X, No. 34, 1928, pp. 466—484.

—— "Studies in the History of Statistical Method." Williams and Wilkins, Baltimore, 1929.

WICKSELL, S. D.: "The Application of Solid Hypergeometrical Series to Frequency Distributions in Space." *Phil. Mag.*, Vol. XXXIV, 1917, pp. 389—394.

WICKSELL, S. D.: "On the Genetic Theory of Frequency." *Ark. f. Mat., Astr. och Fysik*, Bd. XII, No. 20, 1917.

—— "On Logarithmic Correlation, with an Application to the Distribution of Ages at First Marriage." *Sv. Akt. Tidskr.*, Häft. 4, 1917, pp. 1—21.

—— "The Correlation Function of Type A and the Regression of its Characteristics." *Kungl. Sv. Vet. Akad. Handl.*, Bd. LVIII, No. 3, 1917, pp. 1—48.

—— "The Construction of the Curves of Equal Frequency in case of Type A Correlation." *Sv. Akt. Tidskr.*, Häft. 2—3, 1917, pp. 1—19.

—— "Contributions to the Analytical Theory of Sampling." *Ark. f. Mat., Astr. och Fysik*, Bd. XVII, No. 19, 1923.

YULE, G. UDNY: "On the Significance of Bravais' Formulae for Regression, etc., in the case of Skew Correlation." *Roy. Soc. Proc.*, Vol. LX, 1897, pp. 477—489.

THE DERIVATION OF CERTAIN HIGH ORDER SAMPLING PRODUCT MOMENTS FROM A NORMAL POPULATION.

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The recent publication, by R. A. Fisher*, of a paper on the derivation of moments and product moments of sampling distributions has not only brought to a focus the work that has been done in this field by a large number of workers, but has also set these different contributions in their true perspective, and shown them to be partial attempts to deal with the whole matrix of direct and product moments of the various symmetric functions customarily calculated from the sample, and used as presumptive values in defining the population of which the sample is a member. Dr Fisher has supplied all the formulae required up to the 10th degree, together with four others of special interest of the 12th degree. He gave in addition a rule rendering it unnecessary to deal in the above classification with moments involving powers of the first sample moment coefficient, i.e. the mean, a further rule (dealt with in greater detail elsewhere†) for finding, for normal populations, the variance of the rth symmetric function of the observations, as defined by him, and an extension to multivariate distributions. It might seem that these results were sufficient, and that still higher order results, if they could be derived, would be of no practical utility. But at the end of his paper, in a section dealing with measures of departure from normality, Dr Fisher considered a problem requiring new formulae of higher degree in order to determine the second, fourth and sixth semi-invariants of an expression analogous to $\sqrt{\beta_1}$ of the sample, and similar expressions for the equivalent of $\beta_2 - 3$. These expressions are only approximate, being expansions in powers of $1/n$, and proceed as far as terms in $1/n^2$ in the first case and $1/n$ in the second. If it is desired to proceed to a higher degree of approximation in order to test the convergence of the series reached, particularly for the higher semi-invariants, and so to determine how large the sample must be before deductions can safely be drawn as to the normality or otherwise of the population from which the sample has been taken, then further formulae are required. It is the purpose of the present paper to supply the formulae which will enable the results to be pushed to a further stage in the approximation. Their application in this connection has been made by Dr E. S. Pearson in the paper following this one‡.

As a number of different papers have appeared on the subject in the last year or two, it may be well to begin by pointing out their essential differences. Thiele,

* R. A. Fisher, *Proc. Lond. Math. Soc.* (2), Vol. xxx. (1929), pp. 199—238.

† J. Wishart, *Proc. Roy. Soc. Edin.* Vol. xlix. (1929), pp. 78—90.

‡ See pp. 239—249.

in 1889*, after defining the semi-invariants, used symmetric functions of the observations of a sample which are the same functions of the sample moment coefficients as the population semi-invariants are of the population moment coefficients. He supplied an expression covering all the semi-invariants of the mean; the first three semi-invariants of his symmetric function of 2nd degree (i.e. of the variance); the first two semi-invariants of his 3rd and 4th degree expressions; and the first semi-invariant or mean value of his 5th and 6th degree expressions. The steps in the derivation of these formulae were only outlined, the results depending on the use of tables of symmetric functions of the roots of equations; such tables were given by Thiele at the end of his paper. Earlier tables of this kind had been given by Cayley and other writers†. Later, in 1903‡, Thiele added a fourth semi-invariant of the variance, and the mean values of his 7th and 8th degree expressions.

Later Tchouproff§ obtained certain general results for the moment coefficients of moment coefficients, and in particular gave formulae for the first four moment coefficients of the variance; these were also given later by Church‖, by a method which has been described as the method of "Student"¶; it is essentially the same as that employed by Thiele, but later workers were probably unaware that the tables of symmetric functions had already been published in works on pure mathematics. It is clear, as has been recently pointed out by Rider**, that the formulae of Tchouproff for the moment coefficients of the variance can be derived from the earlier results of Thiele by using the general formulae connecting moments and semi-invariants.

In an important recent paper, C. C. Craig dev[illegible]d Thiele's work quite extensively††. It should be remembered that there [illegible]e three kinds of moment coefficients, or semi-invariants, arising in this work. T[illegible]ese are (1) the moment coefficients, or other symmetric functions of the sample observations, which are regarded as estimates of (2), the moment coefficients [illegible]semi-invariants of the population of which the observations form a random sampl[illegible]inally the distribution in all possible random samples of the sample moment coefficients or other symmetric functions is specified by means of (3), its direct and product moment coefficients, or semi-invariants. Both Fisher and Craig postulate an infinite population having any law of distribution with finite moments. Craig supposes the ordinary moments, m_r, to be calculated from the sample, and he gives formulae for the semi-invariants of the multiple distribution of these moments, in terms of the population semi-invariants. His list is not exhaustive, for he considers only the semi-invariants for the simultaneous distributions of the second and third, and second and fourth,

* T. N. Thiele, *Forelæsninger over Almindelig Iagttagelseslære*, Copenhagen, 1889.

† See Salmon's *Higher Algebra*, where the functions are given up to the 10th order.

‡ T. N. Thiele, *Theory of Observations*, London, 1903, C. and E. Layton, pp. 45—46.

§ A. A. Tchouproff, *Biometrika*, Vol. XII. (1919), pp. 140—169 and 185—210.

‖ A. E. R. Church, *Biometrika*, Vol. XVII. (1925), pp. 79—83.

¶ Used by "Student" in calculating the moment coefficients of the mean and variance in samples from a normal population. *Biometrika*, Vol. VI. (1908), pp. 1—25.

** P. R. Rider, *Proc. National Academy of Sciences* (U.S.A.), Vol. XV. (1929), pp. 430—434.

†† C. C. Craig, *Metron*, Vol. VII. (1928), pp. 3—74.

moments. For the normal case, however, he does give three exact expressions ($S_{22}(\nu_2\nu_4)$, $S_{13}(\nu_2\nu_4)$ and $S_{04}(\nu_2\nu_4)$ in his notation) which were not tabulated by Fisher, although the latter, in connection with the tests for normality, gave the leading terms in the formulae corresponding to the last two of these formulae. Further results by Craig are general approximate formulae for the first four semi-invariants of $\sqrt{\beta_1}$, β_2-3 and σ, which are only worked out fully, however, for the normal case. A section of Fisher's paper also deals with this point and furnishes a higher degree of approximation for the semi-invariants of an expression equivalent to $\sqrt{\beta_1}$, adding the sixth (the fifth being zero), while for the equivalent of β_2-3 he goes as far as terms in $1/n^3$ as Craig does, but adds a fifth semi-invariant. I have been unable, however, in the case of β_2 to verify Craig's terms in $1/n^3$, which do not agree with Fisher's*. Important as Craig's contribution to the theory is, however, it should be pointed out that there is an essential difference in Fisher's method, a difference that makes for simplicity in the resulting formulae. Craig at one point remarks "it rather seems that the best hopes of effectively further simplifying the problem of sampling for statistical characteristics lie either in the discovery of a new kind of symmetric function of all the observations which may be used to characterise frequency functions and which will be more amenable than either moments or semi-invariants for use in sampling problems, or in, what may very well prove to be better and more feasible, the abandonment of the method of characterising frequency functions by symmetric functions of all the observations altogether." The line of Fisher's work had followed the first of these two suggestions. The symmetric functions of the observations which he supposes calculated, i.e. his k's, are such that the mean value of any k_r in all possible samples, is κ_r, the rth semi-invariant of the population from which the samples are taken. His k's are, therefore, more nearly allied to semi-invariants than to ordinary moments, and they are not the same functions of the sample moments as ordinarily defined as the semi-invariants are of the population moments. Thus, for example, if we denote by m_r the rth moment coefficient of the sample about its own mean, i.e.

$$m_r=\frac{1}{n}\overset{n}{S}_1(x-\bar{x})^r, \qquad \bar{x}=\frac{1}{n}\overset{n}{S}_1(x),$$

then the first four of Fisher's symmetric functions are

$$k_1=m_1, \quad k_2=\frac{n}{n-1}m_2,$$

$$k_3=\frac{n^2}{(n-1)(n-2)}m_3, \quad k_4=\frac{n^2}{(n-1)(n-2)(n-3)}\{(n+1)m_4-3(n-1)m_2^2\}.$$

The employment of these functions of the ordinary moment coefficients leads to a great simplification, not only of the sampling results, but also of the methods by

* Craig's formula for c_1, the mean value of β_2, is certainly wrong in the terms in $1/n^3$, for the exact result is $3\frac{n-1}{n+1}$, which if expanded gives a term $-1/n^3$ in place of Craig's $-5/n^3$. It may be that Craig's degree of approximation hardly warranted his giving the terms in $1/n^3$ for the semi-invariants of β_2, although the term in $1/n^3$ in κ_2 (Craig's c_2) is correct.

which these are derived. For although Fisher shows by an example how to proceed by direct algebraic methods to determine the semi-invariants of the multiple distribution of powers and products of any number of the k's, expressing the results in terms of the population semi-invariants, he soon makes it clear that the intermediate steps may be left out and the final result written down, a term at a time, by methods of combinatorial analysis, following certain simple rules. The demonstration of the validity of the rules is admittedly a difficult piece of mathematics, but the rules themselves are easy to remember and far simpler to apply than the direct algebraic methods. To illustrate first the nature of the general problem, suppose that we are concerned with the derivation of the formula for $\kappa(3^p 2^q)$. First as to the meaning of this expression; k_2 and k_3 have already been defined in terms of the observations of the sample. If we write $\mu(3^p 2^q)$ for the mean value of $k_3^p k_2^q$ taken over all possible samples, then $\kappa(3^p 2^q)$ is the corresponding semi-invariant, the κ's and μ's being related by the identity in t_2 and t_3,

$$1+\mu(2)\,t_2+\mu(3)\,t_3+\mu(2^2)\frac{t_2^2}{2!}+\mu(23)\,\frac{t_2t_3}{1!\,1!}+\mu(3^2)\,\frac{t_3^2}{2!}+\ldots$$

$$\equiv \exp\left\{\kappa(2)\,t_2+\kappa(3)\,t_3+\kappa(2^2)\,\frac{t_2^2}{2!}+\kappa(23)\,\frac{t_2t_3}{1!\,1!}+\kappa(3^2)\,\frac{t_3^2}{2!}+\ldots\right\}.$$

$\kappa(3^p 2^q)$ may be expressed as the sum of terms each of order $3p+2q$, consisting of powers and products of the semi-invariants of the sampled population, $\kappa_2, \kappa_3, \ldots, \kappa_{3p+2q}$, and the general rules of the combinatorial procedure for determining the coefficients have been given by Fisher*. A simple illustration is given at the end of the present paper.

In the case where the sampled population is normal we see at once the advantage of expressing the results in terms of semi-invariants, for all the κ's above κ_2 vanish, and thus we are left with only a single coefficient to evaluate—in the above example that of $(\kappa_2)^{\frac{1}{2}(3p+2q)}$†. Certain interesting generalisations also follow. Thus the semi-invariants of the distribution of powers and products of moment coefficients, or of k's, of the second order, may be solved by considering appropriate ring arrangements of rods‡, while general formulae for the variance of k_t and for the correlation between product moments of any order have also been determined for the normal case§.

We shall now consider in detail in the case of a normal population the derivation of the formula for $\kappa(3^2 2^3)$ which is one of the new results required for further development of the tests for normality. The result will consist of a single term in κ_2^6, since the expression evaluated is of the 12th degree, κ_2 being simply σ^2, the variance of the sampled population. In following out the rules we must therefore write down all the two-way partitions which have

5 *columns*, containing 3, 3, 2, 2 and 2 entries respectively (controlled by $\kappa(3^2 2^3)$),

6 *rows*, each containing 2 entries (controlled by κ_2^6).

* R. A. Fisher, *loc. cit.* pp. 219—223.

† The coefficient vanishes when p is odd.

‡ J. Wishart, *Proc. Lond. Math. Soc.* (2), Vol. XXIX. (1929), pp. 309—321.

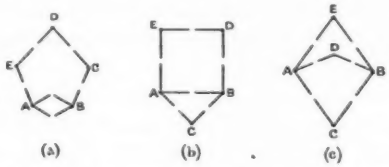
§ J. Wishart, *Proc. Roy. Soc. Edin.* Vol. XLIX. (1929), pp. 78—90.

We find that there are three partitions that have other than zero coefficients, as follows:

1	1	—	—	—	2	1	1	—	—	—	2	1	—	1	—	—	2
1	1	—	—	—	2	1	—	1	—	—	2	1	—	—	1	—	2
1	—	1	—	—	2	1	—	—	—	1	2	1	—	—	—	1	2
—	1	—	1	—	2	—	1	1	—	—	2	—	1	1	—	—	2
—	—	1	—	1	2	—	1	—	1	—	2	—	1	—	1	—	2
—	—	—	1	1	2	—	—	—	1	1	2	—	1	—	—	1	2
3	3	2	2	2	12	3	3	2	2	2	12	3	3	2	2	2	12
			(*a*)						(*b*)						(*c*)				

No row can contain only a single entry, so that no 2 can occur, and the entries are therefore all units*. Now if the columns be regarded as corners having as many arms extending from them as there are units in the column, and if the rows be regarded as the junctions between the arms of different corners, we have the five corners shown in the diagram below to dispose in all possible ways so that there are six junctions

and two arms only at each junction. The partitions (*a*), (*b*) and (*c*) are then found to be equivalent to the symbolical figures



Figures (*a*) and (*b*) may be regarded as two-dimensional models, unless it is preferred to think of the arms connecting A to B, directly and through C, as in a plane perpendicular to the paper. Figure (*c*), on the other hand, is best regarded as a symmetrical figure in three dimensions.

In each case there are two parts to the desired result. One is the numerical coefficient, which is derived from the number of ways in which the particular diagram, or partition, may be set up, the arms of any corner having separate identities and being interchangeable, and the corners themselves being regarded as distinct. The other part is the coefficient in n, the size of the sample, and this is fixed from the nature of the pattern, i.e. the arrangement of entries and zeros, irrespective of what the entries actually are, and the number of rows and columns—alternatively, looking at the symbolical diagrams, the n-coefficient is fixed by the number of corners and the number of breaks between corners and by the design of the diagram. The two parts will be considered separately.

* For the conditions under which certain patterns have zero-coefficients see R. A. Fisher, *loc. cit.* pp. 220—221.

(1) *Numerical coefficient.* This is determined most simply from the symbolical diagram.

(*a*) Having placed the two 3-way corners (A and B) together, so that they can be doubly linked, there are six ways of disposing the three 2-way corners (C, D and E) so as to form a ring. An arm (B to C) may be selected out of the three at B in three ways, likewise for the arm A to E, and in the case of the three 2-arm corners there are two ways each of arranging the arms (C to B or D, etc.). Finally there are two ways of linking the double arms between A and B. The total numerical coefficient is therefore $6 \cdot 3^2 \cdot 2^3 \cdot 2 = 864$.

(*b*) There are three ways of putting a 2-way corner (C, D or E) to one side of the line AB to form a triangle, and two ways of connecting up the others (D or E to B). These give a factor 6. The arms A to E, A to B and A to C may be stretched out in six ways, likewise for the three arms from B, while there are two ways of linking for each of the three 2-arm corners. The numerical coefficient is therefore $6^3 \cdot 2^3 = 1728$.

(*c*) This is a symmetrical arrangement. There are six ways each of stretching out the arms at A and B, and two ways each of linking up to the arms from C, D and E. The numerical coefficient is then $6^2 \cdot 2^3 = 288$.

As a check on the correctness of the total numerical coefficient we note that the total, $864 + 1728 + 288$, is equal to 288×10. 288 is the numerical coefficient of the term in κ_2^5 of $\kappa(3^2 2^2)$ (see R. A. Fisher, *loc. cit.* p. 213, formula (31)), and 10 is the degree of $\kappa(3^2 2^2)$. The coefficients can in fact be determined from the number of ways in which a new corner may be inserted into the pattern of lower degree. In the case of $\kappa(3^2 2^2)$ there are five junctions, and a new 2-way corner can therefore be inserted in five ways, and when its position has been decided, there are two ways of linking up the arms of the new corner to those at the broken junction. Hence the numerical coefficient for $\kappa(3^2 2^3)$, obtained by adding a 2 to $\kappa(3^2 2^2)$, is equal to the number of ways in which the pattern for $\kappa(3^2 2^2)$ can be arranged, namely 288, multiplied by 10. The diagrams on the top of p. 230 show the development from the simple result

$$\kappa(3^2) = \frac{6n}{(n-1)(n-2)} \kappa_2^3$$

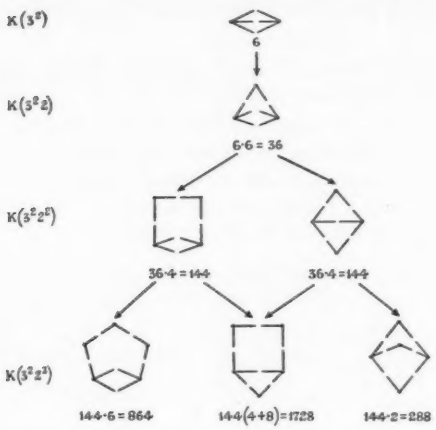
as far as the numerical coefficient is concerned.

The derivation of new formulae can evidently, as to the numerical coefficient, be pushed as far as desired. In fact the *total* numerical coefficient of the term in κ_2^{r+3} of $\kappa(3^2 2^r)$ is

$$2^r \cdot \frac{(r+2)!}{2!} \cdot 6.$$

(2) *The n-coefficient.* It is already known that the term in κ_2^{r+3} of $\kappa(3^2 2^r)$ is of the order $1/n^{r+1}$ so that, approximately,

$$\kappa(3^2 2^r) = \frac{3 \cdot 2^r \cdot (r+2)!}{n^{r+1}} \cdot \kappa_2^{r+3};$$



but if the exact coefficient is wanted, it is necessary to follow out the rules given by Fisher (pp. 221—222), and proved on pp. 226—230. It so happens that our three patterns, although essentially different in their structure, have the same n-coefficient, namely

$$\frac{n}{(n-1)^4\,(n-2)}.$$

This is due to the fact that they are all derived from the same pattern, namely that for $\kappa(3^2)$, by the insertion of fresh 2-way corners. By way of illustration of the general method one only of the above patterns will be evaluated, and this is more conveniently done from the symbolical diagram than from the 2-way partition. It should be noted that this work does not have to be repeated every time an example is worked: when the coefficient has been determined for any pattern it applies to all patterns of that kind, irrespective of the entries. R. A. Fisher has supplied some three pages of the more commonly occurring patterns (pp. 223—226), and it is only because our example is not covered by his list that it requires to be worked out*. We shall choose pattern (a) for illustration. The arms may now be regarded as being joined, thus:

and the reader is recommended to reconstruct this figure for himself by means of six matches, in order to follow more easily the reasoning.

* The reasoning which follows is given for the purpose of illustration, but it is unlikely that a pattern of such a complex nature will ever require to be worked out by the reader. An indication is given later of methods whereby the n-coefficients of such patterns can be derived from those of simpler patterns: e.g. in the present case the coefficient of the normal term of $\kappa(3^2)$ is all that is required to develop the normal term of such a semi-invariant as $\kappa(3^2 2^r)$.

The rods (matches) are now regarded as the rows of the 2-way partition, and we consider all the possible ways in which the rods can be separated into 1, 2, 3, ... 6 separate groups, or separates. With each of these there is associated a factor in n, according to the scheme of the following table, which also shows the number of such separates.

	Separation into	*Number of ways*	*Factor*
(*a*)	1 separate	1	n
(*b*)	2 separates of 1 and 5	6	
	2 separates of 2 and 4	15	
	2 separates of 3 and 3	10	31 $\quad n(n-1)$
(*c*)	3 separates of 1, 1 and 4	15	
	3 separates of 1, 2 and 3	60	
	3 separates of 2, 2 and 2	15	90 $\quad n(n-1)(n-2)$
(*d*)	4 separates of 1, 1, 1 and 3	20	
	4 separates of 1, 1, 2 and 2	45	65 $\quad n(n-1)(n-2)(n-3)$
(*e*)	5 separates of 1, 1, 1, 1 and 2	15	$n(n-1)(n-2)(n-3)(n-4)$
(*f*)	6 separates	1	$n(n-1)(n-2)(n-3)(n-4)(n-5)$

In each of these 203 separations we consider the corners separately. Each unbroken corner contributes a factor n^{-1}, a corner broken into two parts the factor $-\frac{1}{n(n-1)}$, into three parts the factor $\frac{2!}{n(n-1)(n-2)}$. The nature of the separations will be expressed by a quantity in brackets, such as $(1^p 2^q 3^r)$, which specifies p unbroken corners, q broken into two parts and r into three parts, so that for our example $p+q+r=5$ always, and such a group of separates would contribute a term

$$\left(\frac{1}{n}\right)^p \cdot \left(-\frac{1}{n(n-1)}\right)^q \cdot \left(\frac{2!}{n(n-1)(n-2)}\right)^r.$$

(*a*) Here all the corners are unbroken, and the coefficient is n/n^5.

(*b*) Let us number the rods as follows:

(i) Separation into two separates of 1 and 5, however it is done, leaves three corners unbroken, while the other two are broken into two parts. We therefore have a contribution of $6(1^3 2^2)$.

(ii) Separates of 2 and 4. The fifteen ways that this can be done may be divided into a number of sub-classes. Thus if we separate off 1 and 2 from the rest we obtain a term $(1^3 2^2)$ and the same result is obtained by separating off 3 and 4, 4 and 5, or 5 and 6. Total $4(1^3 2^2)$. Separating off 1 and 3 produces $(1^2 2^3)$ and the same is true of 1 and 6, 2 and 3, and 2 and 6. Total $4(1^2 2^3)$. Separating off 1 and 4 is the same as separating 1 and 5, 2 and 4, 2 and 5, 3 and 5, 3 and 6, and 4 and 6. Total $7(1^4)$. The separates of 2 and 4 therefore contribute $4(1^3 2^2) + 4(1^2 2^3) + 7(1^4)$.

(iii) Separates of 3 and 3. This can be done in ten ways, subdivided as follows:

1, 2, 3 and 1, 2, 6	$2(1^3 2^2)$,
1, 2, 4; 1, 2, 5; 1, 3, 6 and 1, 4, 5	$4(1\ 2^4)$,
1, 3, 4 and 1, 5, 6	$2(1^2 2^3)$,
1, 3, 5 and 1, 4, 6	$2(2^5)$.

The total contribution from the 31 separations into 2 separates is therefore

$$12(1^3 2^2) + 6(1^2 2^3) + 11(12^4) + 2(2^5).$$

(*c*) The separations into three separates are a little more difficult to follow out*, but result in

$$1(1^3 3^2) + 10(1^2 2^3) + 12(1^2 2^2 3) + 22(12^4) + 12(12^3 3) + 6(12^2 3^2) + 13(2^5) + 12(2^4 3) + 2(2^3 3^2).$$

(*d*) The separations into four separates lead to

$$3(1^2 2 3^2) + 5(12^4) + 12(12^3 3) + 9(12^2 3^2) + 9(2^5) + 16(2^4 3) + 11(2^3 3^2).$$

(*e*) For five separates we have

$$3(12^2 3^2) + 1(2^5) + 4(2^4 3) + 7(2^3 3^2).$$

(*f*) Here all the corners are broken completely and we have simply $(2^3 3^2)$.

The final n-coefficient is then made up as follows:

$$\frac{1}{n^4}\left[1 + \frac{12}{n-1} - \frac{6}{(n-1)^2} + \frac{11}{(n-1)^3} - \frac{2}{(n-1)^4} + \frac{4}{(n-1)(n-2)} - \frac{10(n-2)}{(n-1)^2} + \frac{24}{(n-1)^2}\right.$$
$$+ \frac{22(n-2)}{(n-1)^3} - \frac{24}{(n-1)^3} + \frac{24}{(n-1)^3(n-2)} - \frac{13(n-2)}{(n-1)^4} + \frac{24}{(n-1)^4} - \frac{8}{(n-1)^4(n-2)}$$
$$- \frac{12(n-3)}{(n-1)^2(n-2)} + \frac{5(n-2)(n-3)}{(n-1)^3} - \frac{24(n-3)}{(n-1)^3} + \frac{36(n-3)}{(n-1)^3(n-2)} - \frac{9(n-2)(n-3)}{(n-1)^4}$$
$$+ \frac{32(n-3)}{(n-1)^4} - \frac{44(n-3)}{(n-1)^4(n-2)} + \frac{12(n-3)(n-4)}{(n-1)^3(n-2)} - \frac{(n-2)(n-3)(n-4)}{(n-1)^4}$$
$$\left. + \frac{8(n-3)(n-4)}{(n-1)^4} - \frac{28(n-3)(n-4)}{(n-1)^4(n-2)} - \frac{4(n-3)(n-4)(n-5)}{(n-1)^4(n-2)}\right].$$

This reduces to

$$\frac{n}{(n-1)^4(n-2)}.$$

As already stated all three patterns for the contributory portions of $\kappa(3^2 2^3)$ have the same n-coefficient, and our result in full, therefore, is *for normality*

$$\kappa(3^2 2^3) = \frac{2880n}{(n-1)^4(n-2)}\kappa_2^6 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

* (*c*), (*d*) and (*e*) are best evaluated at the same time as (*b*), to avoid repetition of labour.

List of Higher Order Formulae.

The following results, which are of degree 12 and upwards, will enable expansions for the moments of $\sqrt{\beta_1}$ and β_2 from a normal population to be determined to a higher degree of approximation than has hitherto been reached.

$$\kappa(3^2 2^4) = \frac{34560n}{(n-1)^5(n-2)}\kappa_2^7 \quad \text{.....(2)}, \qquad \kappa(3^4 2) = \frac{7776n^2(5n-12)}{(n-1)^4(n-2)^3}\kappa_2^7 \quad \text{.....(3)},$$

$$\kappa(3^4 2^2) = \frac{108864n^2(5n-12)}{(n-1)^5(n-2)^3}\kappa_2^8 \quad \text{.....(4)},$$

$$\kappa(3^4 2^3) = \frac{1741824n^2(5n-12)}{(n-1)^6(n-2)^3}\kappa_2^9 \quad \text{.....(5)},$$

$$\kappa(3^6) = \frac{466560n^3(22n^2-111n+142)}{(n-1)^5(n-2)^5}\kappa_2^9 \quad \text{.....(6)},$$

$$\kappa(3^6 2) = \frac{18}{n-1}\kappa_2\kappa(3^6) \quad \text{.....(7)}, \qquad \kappa(3^6 2^2) = \frac{360}{(n-1)^2}\kappa_2^2\kappa(3^6) \quad \text{.....(8)},$$

$$\kappa(4^2 2^2) = \frac{1920n(n+1)}{(n-1)^3(n-2)(n-3)}\kappa_2^6 \quad \text{.....(9)},$$

$$\kappa(4^2 2^3) = \frac{23040n(n+1)}{(n-1)^4(n-2)(n-3)}\kappa_2^7 \quad \text{.....(10)},$$

$$\kappa(4^2 2^4) = \frac{322560n(n+1)}{(n-1)^5(n-2)(n-3)}\kappa_2^8 \quad \text{.....(11)},$$

$$\kappa(4^3 2) = \frac{20736n(n+1)(n^2-5n+2)}{(n-1)^3(n-2)^2(n-3)^2}\kappa_2^7 \quad \text{.....(12)},$$

$$\kappa(4^3 2^2) = \frac{290304n(n+1)(n^2-5n+2)}{(n-1)^4(n-2)^2(n-3)^2}\kappa_2^8 \quad \text{.....(13)},$$

$$\kappa(4^3 2^3) = \frac{4644864n(n+1)(n^2-5n+2)}{(n-1)^5(n-2)^2(n-3)^2}\kappa_2^9 \quad \text{.....(14)},$$

$$\kappa(4^4) = \frac{6912n(n+1)}{(n-1)^3(n-2)^3(n-3)^3}\{53n^4-428n^3+1025n^2-474n+180\}\kappa_2^8 \quad \text{.....(15)},$$

$$\kappa(4^4 2) = \frac{16}{n-1}\kappa_2\kappa(4^4) \quad \text{.....(16)}, \qquad \kappa(4^4 2^2) = \frac{288}{(n-1)^2}\kappa_2^2\kappa(4^4) \quad \text{.....(17)},$$

$$\kappa(4^5) = \frac{364 \cdot 12^5}{n^4}\kappa_2^{10}, \text{ approximately}^* \quad \text{.....(18)}.$$

The derivation of the foregoing results has been rendered much simpler than it would otherwise have been by the discovery of a rule which applies whenever a fresh k_2 is introduced into the kappa expression to be evaluated. We may regard $\kappa(3^2)$, $\kappa(3^2 2)$, ... $\kappa(3^2 2^r)$ for example as a train of formulae each derived from the preceding one by the adding of a k_2, or, looking at the symbolical diagram, by the insertion of

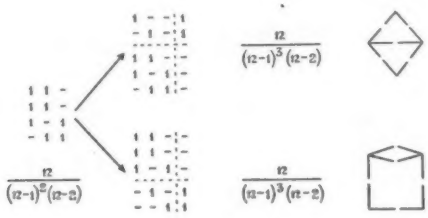
* There is a misprint in Dr Fisher's paper, p. 236, where $\kappa_3(x)$ (not κ_6 as printed) should read $71 \cdot 144 \cdot \sqrt{6} \cdot n^{-\frac{5}{2}}$. Also on p. 233 (top) the three ways of evaluating the symbolical diagrams for $\kappa(3^4 2)$ should be 15552, 7776 and 15552 respectively.

a fresh 2-way corner. The effect this has on the numerical coefficient of the normal term in $\kappa(3^2 2^r)$ is to multiply by the degree of this expression, i.e. by $2r+6$, in order to produce the normal term of $\kappa(3^2 2^{r+1})$, as explained on p. 229. The effect on the n-coefficient is simply to divide by $n-1$ every time. For example, the normal term in $\kappa(3^2 2)$ is derived from that of $\kappa(3^2)$ as follows:

1	1		1	1	–
1	1		1	1	–
1	1		1	–	1
			–	1	1

$$\frac{n}{(n-1)(n-2)}, \qquad \frac{n}{(n-1)^2(n-2)}.$$

One of the rows (here the last) must be split to make two new rows, and the units of the new column placed in these rows. Now if in the working out of the n-coefficient of the pattern for the normal term of $\kappa(3^2 2)$ the separations are grouped into two classes, (1) those in which the last two rows occur together and therefore reproduce the separations of the normal term of $\kappa(3^2)$ together with the contribution $\frac{1}{n}$ from the new column, and (2) those in which the last two rows are separated, bringing in a contribution $-\frac{1}{n(n-1)}$ from the new column, it may be readily verified that the net effect on the coefficient of the normal term of $\kappa(3^2)$ is to divide by $n-1$. In the next stage two patterns are produced for $\kappa(3^2 2^2)$ corresponding with the symbolical diagrams on p. 230, according as one of the first two or one of the last two rows of the pattern for $\kappa(3^2 2)$ is split to form two new rows. Thus:



The n-coefficients are equal, both being derived from that of $\kappa(3^2 2)$ by dividing by $n-1$.

In the last stage of the process for the example illustrated, i.e. $\kappa(3^2 2^3)$, three new patterns are formed from the two patterns of $\kappa(3^2 2^2)$ as shown on p. 230, and their n-coefficients are all equal, and equal to $\frac{n}{(n-1)^4(n-2)}$. We are now able to write down the general formula

$$\kappa(p^q 2^r) = \frac{2^r (r+\frac{1}{2}pq-1)!}{(\frac{1}{2}pq-1)!(n-1)^r}\, \kappa_2{}^r \,.\, \kappa(p^q) \quad \text{.....(19),}$$

which is generally true for sampling from a normal population, for it holds when pq is even, while when pq is odd the whole expression vanishes, since $\kappa(p^q)$ has in general no term involving κ_2 only, when pq is odd. A special case of (19) occurs when $q=1$. We then find that, for the normal case,

$$\kappa(p\,2^r)=\frac{2^r(r+\tfrac{1}{2}p-1)!}{(\tfrac{1}{2}p-1)!\,(n-1)^r}\,\kappa_2{}^r\,.\,\kappa(p)=0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(20),$$

when p is greater than 2. This follows from the general result that $\kappa(p)=\kappa_p$, while all κ's above κ_2 vanish for normality. It is obvious also from the impossibility of constructing a closed symbolical diagram to fit this case; for, to illustrate from $\kappa(p\,2^2)$, the only possible diagram is of the form

the number of arms extending from A being equal to p, and the conditions laid down by Fisher (*loc. cit.* pp. 220—221) are such that (1) no loose arms can exist and (2) a break at any one corner must not divide the figure into two separate pieces.

In the special case where $p=2$ we have

$$\kappa(2^{r+1})=\frac{2^r\,.\,r!}{(n-1)^r}\,\kappa_2{}^{r+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(21),$$

which is the $(r+1)$th semi-invariant of the distribution of k_2, in samples from a normal distribution. In terms of the more familiar m_2, the second moment coefficient of the sample, the result is

$$\frac{2^r\,.\,r!\,(n-1)}{n^{r+1}}\,.\,\sigma^{2(r+1)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(22),$$

a form which has already been published *, and which is derivable from "Student's" distribution of the variance †, σ being the standard deviation of the sampled normal population. For $p>2$ equation (20) shows that there can be no correlation in samples between k_p and any power of k_2. For comparison with this we have a more general result already reached, to the effect that no correlation can exist between k_t and k_u unless $t=u$, or, for bi-variate populations, between k_{tu} and k_{vw} unless $t+u=v+w$. This is another important property of the k functions which does not hold among moments. For correlation does exist between the sample moment coefficients of different orders, other than m_1, the mean, which is uncorrelated with any of the higher moments.

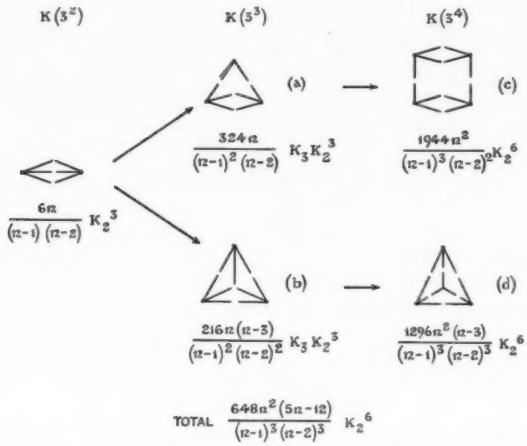
The tabulated formulae (1) to (18) are special cases of the general result (19) except for $\kappa(3^6)$, $\kappa(4^4)$ and $\kappa(4^5)$, and it is evident that any expression of the form $\kappa(p^q\,2^r)$ can be evaluated in full for the normal case as long as $\kappa(p^q)$ is known exactly. The numerical parts of $\kappa(3^6)$, $\kappa(4^4)$ and $\kappa(4^5)$ have already been worked out by R. A. Fisher‡, although he did not in his paper give the separate contribu-

* J. Wishart, *Proc. Lond. Math. Soc.* (2), Vol. XXIX. (1929), p. 314.

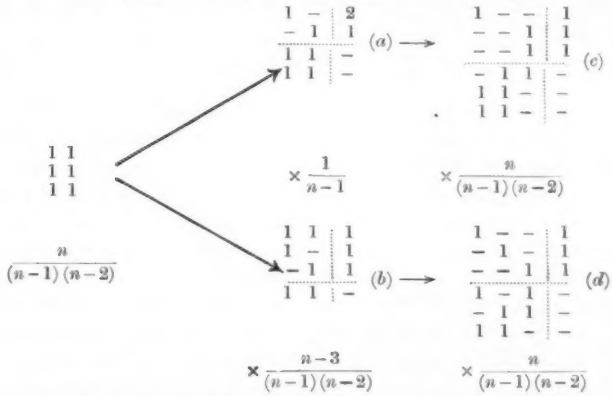
† "Student," *Biometrika*, Vol. VI. (1908), pp. 6—8.

‡ R. A. Fisher, *loc. cit.* pp. 233 and 236.

tions of the different patterns. An extension of the rule already described for the adding of a k_2 enables us to develop the n-coefficients of the patterns in $\kappa(3^6)$, e.g. from those of $\kappa(3^4)$, which are known*. By way of illustration we give the development of the two patterns for the normal term of $\kappa(3^4)$ from the single pattern for $\kappa(3^2)$.



Alternatively the development in terms of the 2-way partitions is as follows:



(*a*) The rule here is that already described, for the new k_3 added is in the form of a column having only two entries. The n-coefficient is therefore obtained from that of $\kappa(3^2)$ by dividing by $n-1$, while its numerical multiplier is multiplied by 54. (Three ways of breaking the old pattern—six ways of arranging the new 3-way corner at this break—and, since the resulting pattern is unsymmetrical, three ways of choosing which of the 3-way corners shall be unlike the others.)

* These rules will be described in a forthcoming paper by R. A. Fisher and the present author.

(*b*) Consideration of the separations shows that the n-coefficient of this pattern is of the form

$$\frac{n}{(n-1)(n-2)}A - \frac{1}{(n-1)(n-2)}B,$$

where A is the n-coefficient of the 2 row—2 column partition, i.e. $\frac{1}{n-1}$, and B is the n-coefficient of the pattern for the normal term of $\kappa(3^2)$, i.e. $\frac{n}{(n-1)(n-2)}$. The coefficient multiplier is therefore $\frac{n-3}{(n-1)(n-2)}$, while the numerical multiplier is 36. (Three ways of breaking old pattern—two junctions to either of which one of the three arms of the new corner may be connected, while the remaining two arms may be disposed in two ways at the break.)

(*c*) and (*d*) In both these cases there is only one junction, i.e. that where three arms meet, that can be broken if the normal term of $\kappa(3^4)$ is to be formed from the term in $\kappa_3\kappa_2^3$ of $\kappa(3^3)$. The new 3-way corner can then be disposed at this break in six ways, so that the numerical multiplier in (*c*) and (*d*) is 6, while the multiplier of the n-coefficients of the patterns for $\kappa(3^3)$ is in each case $\frac{n}{(n-1)(n-2)}$.

Patterns of $\kappa(3^6)$

Numerical multiplier

699840 2099520 4199040 2799360 466560

n-coefficient

$$\frac{n^3}{(n-1)^5(n-2)^3}, \quad \frac{n^3}{(n-1)^5(n-2)^3}, \quad \frac{n^3(n-3)}{(n-1)^5(n-2)^4}, \quad \frac{n^3(n-3)^2}{(n-1)^5(n-2)^5}, \quad \frac{n^3(n^2-6n+10)}{(n-1)^5(n-2)^5}.$$

$$\text{Total } \frac{466560\, n^3(22n^3-111n+142)}{(n-1)^5(n-2)^5}\kappa_2^9.$$

Patterns of $\kappa(4^4)$

Numerical multiplier

62208 248832 55296

n-coefficient

$$\frac{n(n+1)(n^4-8n^3+21n^2-14n+4)}{(n-1)^3(n-2)^3(n-3)^3}, \quad \frac{n(n+1)(n^4-9n^3+23n^2-11n+4)}{(n-1)^3(n-2)^3(n-3)^3}, \quad \frac{n^2(n+1)^2}{(n-1)^3(n-2)^2(n-3)^2}.$$

$$\text{Total } \frac{6912\, n(n+1)}{(n-1)^3(n-2)^3(n-3)^3}\{53n^4-428n^3+1025n^2-474n+180\}\kappa_2^8.$$

Patterns of $\kappa(4^5)$

Numerical multiplier $12^5.12$ $12^5.80$ $12^5.120$ $12^5.32$ $12^5.120$

Total $\frac{364.12^5}{n^4}\kappa_2^{10}$, approximately.

Procedure in the non-normal case.

In order to fix this simplified problem, which arises in the case of normality, in its place as a part of a wider scheme, it will perhaps be of value to conclude by indicating in a simple example the lines of procedure to be followed in the general case where the sampled population is not normal. Consider $\kappa(3^2)$, the second semi-invariant, or second moment coefficient about the mean, of k_3 in samples. The method of expressing this quantity in terms of the population semi-invariants may be illustrated as follows:

(1) $a_1\,\kappa_6 + a_2\,\kappa_4\,\kappa_2 + a_3\,\kappa_3^2 + a_4\,\kappa_2^3$

(2)

$a_1\,\kappa_6$	$a_2\,\kappa_4\,\kappa_2$	$a_3\,\kappa_3^2$	$a_4\,\kappa_2^3$
3 3 | 6	2 2 | 4	2 1 | 3	1 1 | 2
3 3 |	1 1 | 2	1 2 | 3	1 1 | 2
	3 3 |	3 3 |	1 1 | 2
			3 3 |

(3)

1	9	9	6

(4)

$\frac{1}{n}$	$\frac{1}{n-1}$	$\frac{1}{n-1}$	$\frac{n}{(n-1)(n-2)}$

(5) $$\frac{1}{n}\kappa_6 + \frac{9}{n-1}\kappa_4\kappa_2 + \frac{9}{n-1}\kappa_3^2 + \frac{6n}{(n-1)(n-2)}\kappa_2^3.$$

(1) The result will be a sum of terms containing all the possible 6th order powers or products of the κ's ($\kappa_1 = 0$). There are four such terms.

(2) The 2-way partitions associated with each term are shown in this line. As we are dealing with $\kappa(3^2)$ we shall in each case have 2 columns with contents totalling 3. The number of rows and their contents vary but the rule is simple; for κ_6 one row containing 6; for $\kappa_4\kappa_2$ one row containing 4 and one containing 2; for κ_3^2 two rows each containing 3; for κ_2^3 three rows each containing 2. In this example there is only one possible partition associated with each coefficient (contrasted with the case of $\kappa(3^2 2^3)$ dealt with above) and all the cells are filled.

(3) The numerical coefficients are determined by considering in each case the number of ways of connecting up 2 junctions each having 3 arms. For a_1, a_2, a_3, and a_4 we must make connections of 6, of 4 and 2, of 3 and 3, and of 2, 2 and 2 respectively. It will be seen that the ways in which this can be done are 1, 9, 9 and 6 respectively.

(4) The n-coefficients depend upon the pattern of the four partitions in line (2) above*. These are given by Fisher (*loc. cit.* pp. 223—224), and are set out in line (4) above. The final result obtained by combining the expressions is shown in line (5). In the case where the sampled population is normal we are concerned only with securing the last term.

We have taken of course a very straightforward example in which only simple patterns and rod combinations are required, but the elegance of the method, once it has been grasped, can hardly fail to attract the worker even in far more complex problems.

* For example such partitions as $\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix}$, $\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}$, $\begin{vmatrix} 2 & 2 \\ 2 & 2 \end{vmatrix}$ etc. are all associated with the *same* "pattern," viz. $\begin{vmatrix} \times & \times \\ \times & \times \end{vmatrix}$.

A FURTHER DEVELOPMENT OF TESTS FOR NORMALITY.

By E. S. PEARSON, D.Sc.

(1) Many of the simpler methods of statistical analysis have been developed only for variables which are normally distributed. We have often *a priori* reasons based perhaps on parallel experience for believing that the material is so distributed, but in many cases it is important to obtain evidence on this point from the data, that is to say, it is necessary to apply some test for normality to the sample. The problem is of course two-sided; it is not enough to know that the sample *could* have come from a normal population; we must be clear that it is at the same time improbable that it has come from a population differing so much from the normal as to invalidate the use of "normal theory" tests in further handling of the material. When dealing with a single variable a knowledge of the sampling distribution of β_1 (or $\sqrt{\beta_1}$) and β_2 in terms of the population moments would go far towards the solution of the problem. It is true that there have long been available tables giving the standard errors of β_1 and β_2 and their intercorrelation in terms of the population β_1 and β_2*, but these are based upon the first order terms only in an expansion, and no precise information has been available regarding the size of sample for which the expressions may be considered as accurate, nor as to the shape of the sampling curves. Recent work of Dr C. C. Craig† and Dr R. A. Fisher‡ has, however, now made possible a considerable advance towards the solution of one side of the problem; that is to say, towards a knowledge of the sampling distributions of $\sqrt{\beta_1}$ and β_2 if the population be in fact normal. Reference has been made in the preceding paper to these two sets of results§; in order, however, to place the test on firmer ground and present it in form readily available for practical use, it seemed desirable to carry the expansions for the moments of $\sqrt{\beta_1}$ and β_2 to a higher order of approximation than was reached by these authors. The fresh expressions for the higher semi-invariants given by Dr Wishart have made this extension possible. It should be remembered that as in sampling from a normal population $\sqrt{\beta_1}$ and β_2 are uncorrelated, we have two separate tests. When dealing with other populations it becomes necessary to consider the co-variation of $\sqrt{\beta_1}$ and β_2.

* *Tables for Statisticians and Biometricians*, Part I. Cambridge University Press. Tables XXXVII—XXXIX. The values are of course based on the assumption that the population distribution may be represented approximately by one of the Pearson system of frequency curves.

† C. C. Craig, *Metron*, Vol. VII. 4 (1928), pp. 3—74.

‡ R. A. Fisher, *Proc. Lond. Math. Soc.* (2), Vol. XXX. (1929), pp. 199—238.

§ J. Wishart, pp. 224—238 of the present number of this Journal.

The present paper falls naturally into two parts:

(*a*) In which the results of Fisher and of Wishart have been used to obtain values of the first four moment coefficients of the sampling distributions of $\sqrt{\beta_1}$ and β_2 as far as the terms in n^{-3}.

(*b*) In which tables are given showing for different sizes of sample, n, the values of β_1 and β_2 corresponding to ·05 and ·01 probability points*; these being based on the assumption that Pearson Type VII and Type IV curves with the correct first four moment coefficients will adequately represent the true sampling distributions of the constants as far as these two levels of probability are concerned.

(2) *Moment Coefficients of the Sampling Distribution of* $\sqrt{\beta_1}$.

I shall use here the notation of Fisher and take†

$$x = \sqrt{\frac{(n-1)(n-2)}{6n}}\,\frac{k_3}{k_2^{\frac{3}{2}}} = \frac{n-1}{\sqrt{6(n-2)}}\,\sqrt{\beta_1} \quad \text{.....(1),}$$

$$= \sqrt{\frac{(n-1)(n-2)}{6n}}\,\frac{1}{\kappa_2^{\frac{3}{2}}}\,k_3\left\{1+\frac{k_2-\kappa_2}{\kappa_2}\right\}^{-\frac{3}{2}} \quad \text{.....(2).}$$

Remembering that the population is symmetrical, it is clear that in sampling x will be symmetrically distributed about zero. Let us find the 2nd and 4th moment coefficients of x. We obtain from (2),

$$x^2 = \frac{(n-1)(n-2)}{6n}\,\frac{1}{\kappa_2^3}\left\{k_3^2 - \frac{3}{\kappa_2}k_3^2(k_2-\kappa_2) + \frac{6}{\kappa_2^2}k_3^2(k_2-\kappa_2)^2 - \frac{10}{\kappa_2^3}k_3^2(k_2-\kappa_2)^3 + \frac{15}{\kappa_2^4}k_3^2(k_2-\kappa_2)^4 - \frac{21}{\kappa_2^5}k_3^2(k_2-\kappa_2)^5 + \frac{28}{\kappa_2^6}k_3^2(k_2-\kappa_2)^6 - \ldots\right\} \quad \text{.....(3),}$$

$$x^4 = \frac{(n-1)^2(n-2)^2}{36n^2}\,\frac{1}{\kappa_2^6}\left\{k_3^4 - \frac{6}{\kappa_2}k_3^4(k_2-\kappa_2) + \frac{21}{\kappa_2^2}k_3^4(k_2-\kappa_2)^2 - \frac{56}{\kappa_2^3}k_3^4(k_2-\kappa_2)^3 + \frac{126}{\kappa_2^4}k_3^4(k_2-\kappa_2)^4 - \frac{252}{\kappa_2^5}k_3^4(k_2-\kappa_2)^5 + \frac{462}{\kappa_2^6}k_3^4(k_2-\kappa_2)^6 - \ldots\right\} \quad \text{.....(4).}$$

We must now take mean values of both sides of the equations, and shall need to evaluate terms of the form $\mu(k_3^2, (k_2-\kappa_2)^q)$ and $\mu(k_3^4, (k_2-\kappa_2)^q)$. Following the method of Fisher, these μ's must now be expressed in terms of the corresponding semi-invariants of k_2 and k_3. If u and v be two variables this process is, in general, carried out by means of the identity in t_1 and t_2,

$$1 + \mu(u)t_1 + \mu(v)t_2 + \mu(u^2)\frac{t_1^2}{2!} + \mu(uv)\frac{t_1t_2}{1!\,1!} + \mu(v^2)\frac{t_2^2}{2!} + \ldots$$

$$\equiv \exp\left\{\kappa(u)t_1 + \kappa(v)t_2 + \kappa(u^2)\frac{t_1^2}{2!} + \kappa(uv)\frac{t_1t_2}{1!\,1!} + \kappa(v^2)\frac{t_2^2}{2!} + \ldots\right\} \quad \text{... (5).}$$

* That is to say, the points at which ordinates of the sampling distribution cut off tail areas measuring 5 °/₀ and 1 °/₀ of the total area under the frequency curve.

† The relations between k_2, k_3 and k_4 and the sample moment coefficients have been given above on p. 226 by Wishart. x is really the ratio of m_3 (or k_3) to a sample estimate of its standard error.

In the present case $u = k_3$, $v = k_2 - \kappa_2$; it follows that

(*a*) $\mu(u) = 0 = \mu(v)$; hence $\kappa(u) = 0 = \kappa(v)$.

(*b*) The κ's of 2nd and higher order being independent of the origin chosen,

$$\kappa(k_3^p, (k_2 - \kappa_2)^q) = \kappa(k_3^p, k_2^q) = (3^p 2^q) \text{ (for convenience in writing).}$$

(*c*) Since the population is normal $\kappa(k_3, k_2^q) = (32^q) = 0$ (see Wishart's equation (20), p. 235 above).

Bearing these results in mind and retaining *within* the square brackets of (6) and (7) below terms up to the order of (n^{-4}) for $\mu_2(x)$ and (n^{-5}) for $\mu_4(x)$*, it will be found that

$$\begin{aligned}\mu(x^2) = \mu_2(x) = \frac{(n-1)(n-2)}{6n}\frac{1}{\kappa_2^3}\Big\{&(3^2) - \frac{3}{\kappa_2}(3^2 2)\\ &+ \frac{6}{\kappa_2^2}[(3^2 2^2) + (3^2)(2^2)] - \frac{10}{\kappa_2^3}[(3^2 2^3) + 3(3^2 2)(2^2) + (3^2)(2^3)]\\ &+ \frac{15}{\kappa_2^4}[6(3^2 2^2)(2^2) + 4(3^2 2)(2^3) + (3^2)(2^4) + 3(3^2)(2^2)^2]\\ &- \frac{21}{\kappa_2^5}[15(3^2 2)(2^2)^2 + 10(3^2)(2^3)(2^2)] + \frac{28}{\kappa_2^6}[15(3^2)(2^2)^3]\ldots\Big\} \quad \ldots\ldots\ldots\ldots(6),\end{aligned}$$

$$\begin{aligned}\mu(x^4) = \mu_4(x) = \frac{(n-1)^2(n-2)^2}{36n^2}\frac{1}{\kappa_2^6}\Big\{&(3^4) + 3(3^2)^2 - \frac{6}{\kappa_2}[(3^4 2) + 6(3^2 2)(3^2)]\\ &+ \frac{21}{\kappa_2^2}[(3^4 2^2) + (3^4)(2^2) + 6(3^2 2^2)(3^2) + 6(3^2 2)^2 + 3(3^2)^2(2^2)]\\ &- \frac{56}{\kappa_2^3}[3(3^4 2)(2^2) + (3^4)(2^3) + 6(3^2 2^3)(3^2) + 18(3^2 2^2)(3^2 2)\\ &\qquad + 18(3^2 2)(3^2)(2^2) + 3(3^2)^2(2^3)]\\ &+ \frac{126}{\kappa_2^4}[3(3^4)(2^2)^2 + 36(3^2 2^2)(3^2)(2^2) + 36(3^2 2)^2(2^2) + 24(3^2 2)(3^2)(2^3)\\ &\qquad + 3(3^2)^2(2^4) + 9(3^2)^2(2^2)^2]\\ &- \frac{252}{\kappa_2^5}[90(3^2 2)(3^2)(2^2)^2 + 30(3^2)^2(2^3)(2^2)] + \frac{462}{\kappa_2^6}[45(3^2)^2(2^2)^3]\ldots\Big\} \ldots(7).\end{aligned}$$

We may now substitute into (6) and (7) the expressions for the semi-invariants of k_2 and k_3 tabled by Fisher and Wishart†. Since the population is normal we are concerned with the terms which contain powers of κ_2 only, the population variance; as is necessarily the case, since x is independent of scale the powers of κ_2 divide out and we are left with the following expressions in n, for the 2nd and 4th moment coefficients of x,

$$\begin{aligned}\mu_2(x) &= 1 - \frac{6}{n-1} + \frac{28}{(n-1)^2} - \frac{120}{(n-1)^3} + \ldots\\ &= 1 - \frac{6}{n} + \frac{22}{n^2} - \frac{70}{n^3} + \ldots \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8),\end{aligned}$$

* The term $\kappa(3^p 2^q)$ is of order $n^{-(p+q-1)}$.

† Fisher, *loc. cit.* pp. 210—214; these results are general, for any population. Wishart, p. 233 above; these are for a normal population only.

$$\mu_4(x) = 3 - \frac{90}{n-1} + \frac{1680}{(n-1)^2} - \frac{25{,}200}{(n-1)^3} + \ldots + \frac{5n-12}{n-2}\left(\frac{18}{n-1} - \frac{540}{(n-1)^2} + \frac{10{,}080}{(n-1)^3} - \ldots\right)$$

$$= 3 - \frac{1056}{n^2} + \frac{24{,}132}{n^3} - \ldots \quad \ldots\ldots\ldots\

$$\mu_4(x) = \left\{\frac{(n-1)(n-2)(n-3)}{24n(n+1)}\right\}^2 \frac{1}{\kappa_2^4}\left\{(4^4) + 3(4^2)^2 - \frac{8}{\kappa_2}[(4^4 2) + 6(4^2 2)(4^2)]\right.$$
$$+ \frac{36}{\kappa_2^2}[(4^4 2^2) + (4^4)(2^2) + 6(4^2 2^2)(4^2) + 6(4^2 2)^2 + 3(4^2)^2(2^2)]$$
$$- \frac{120}{\kappa_2^3}[3(4^4 2)(2^2) + (4^4)(2^3) + 6(4^2 2^3)(4^2) + 18(4^2 2^2)(4^2 2)$$
$$+ 18(4^2 2)(4^2)(2^2) + 3(4^2)^2(2^3)]$$
$$+ \frac{330}{\kappa_2^4}[3(4^4)(2^2)^2 + 36(4^2 2^2)(4^2)(2^2) + 24(4^2 2)(4^2)(2^3) + 36(4^2 2)^2(2^2)$$
$$+ 3(4^2)^2(2^4) + 9(4^2)^2(2^2)^2]$$
$$\left. - \frac{792}{\kappa_2^5}[90(4^2 2)(4^2)(2^2)^2 + 30(4^2)^2(2^3)(2^2)] + \frac{1716}{\kappa_2^6}[45(4^2)^2(2^2)^3]\ldots\right\} \quad (16).$$

The values of the semi-invariants of k_4 and k_2 taken from the tables of Fisher and Wishart must now be substituted into (14), (15) and (16). If this is carried out, it is found after reduction that

$$\mu_2(x) = 1 - \frac{12}{n-1} + \frac{100}{(n-1)^2} - \frac{720}{(n-1)^3} + \ldots$$
$$= 1 - \frac{12}{n} + \frac{88}{n^2} - \frac{532}{n^3} + \ldots \quad \ldots\ldots\ldots\

(4) *Approximations to the Probability Integrals of $\sqrt{\beta_1}$ and β_2.*

The first problem to consider is the degree of convergence of the series (10) and (11), and (21), (22) and (23). For this purpose the Tables I and II have been prepared in which the values entered for σ, B_1 and B_2 are based of course only on those terms of the expansions given above. It is necessary to assume that the adequacy of the convergence can be judged from the first four terms of each series. The expressions for the standard errors are clearly adequately represented by the series at $n = 50$. Columns have been inserted showing the degree of approximation

TABLE I. *Moment Coefficients of $\sqrt{\beta_1}$.*

n	$\sqrt{6/n}$	Terms in $\sigma_{\sqrt{\beta_1}}$	$\sigma_{\sqrt{\beta_1}}$	Terms in $B_2(\sqrt{\beta_1})$	$B_2(\sqrt{\beta_1})$
50	·3464	1 − ·060 000 + ·002 400 − ·000 120	·3264	3 + ·720 000 − ·345 600 + ·096 768	3·4712
75	·2828	1 − ·040 000 + ·001 067 − ·000 036	·2718	3 + ·480 000 − ·153 600 + ·028 668	3·3551
100	·2449	1 − ·030 000 + ·000 600 − ·000 015	·2377	3 + ·360 000 − ·086 400 + ·012 096	3·2857
150	·2000	—	·1961	3 + ·240 000 − ·038 400 + ·003 580	3·2052
200	·1732	—	·1706	3 + ·180 000 − ·021 600 + ·001 512	3·1599
250	·1549	—	·1531	3 + ·144 000 − ·013 824 + ·000 774	3·1310
500	·1095	—	·1089	3 + ·072 000 − ·003 456 + ·000 097	3·0686
1000	·0775	—	·0772	3 + ·036 000 − ·000 864 + ·000 012	3·0351

TABLE II. *Moment Coefficients of β_2.*

n	$\sqrt{24/n}$	Terms in σ_{β_2}	σ_{β_2}	Terms in $B_1(\beta_2)$	$B_1(\beta_2)$
50	·6928	1 − ·150 000 + ·013 550 − ·001 159	·5975	1 − ·580 000 + ·207 600 − ·061 096	2·4473
75	·5657	1 − ·100 000 + ·006 022 − ·000 344	·5123	1 − ·386 667 + ·092 267 − ·018 100	1·9800
100	·4899	1 − ·075 000 + ·003 387 − ·000 145	·4547	1 − ·290 000 + ·051 900 − ·007 637	1·6292
150	·4000	1 − ·050 000 + ·001 506 − ·000 043	·3806	1 − ·193 333 + ·023 067 − ·002 261	1·1916
200	·3464	1 − ·037 500 + ·000 847 − ·000 018	·3337	1 − ·145 000 + ·012 975 − ·000 955	·9364
250	·3098	—	·3007	1 − ·116 000 + ·008 304 − ·000 489	·7705
500	·2191	—	·2158	1 − ·058 000 + ·002 076 − ·000 061	·4078
1000	·1549	—	·1538	1 − ·029 000 + ·000 519 − ·000 008	·2098

n	Terms in $B_2(\beta_2)$	$B_2(\beta_2)$
50	3 + 10·800 000 − 8·078 400 + 3·763 296	—
75	3 + 7·200 000 − 3·590 404 + 1·114 876	—
100	3 + 5·400 000 − 2·019 600 + ·470 412	6·8508
150	3 + 3·600 000 − ·897 591 + ·139 383	5·8418
200	3 + 2·700 000 − ·504 900 + ·058 801	5·2539
250	3 + 2·160 000 − ·323 136 + ·030 106	4·8670
500	3 + 1·080 000 − ·080 784 + ·003 763	4·0030
1000	3 + ·540 000 − ·020 196 + ·000 470	3·5203

of $\sqrt{6/n}$ to $\sigma_{\sqrt{\beta_1}}$ and of $\sqrt{24/n}$ to σ_{β_2}*. It may be said roughly that for most practical purposes a knowledge of B_1 and B_2 correct to the 2nd decimal place is sufficient. Thus at $n=100$ and perhaps at 75, the series for $B_2(\sqrt{\beta_1})$ and $B_1(\beta_2)$ may be considered as satisfactory. The convergence of the expression for $B_2(\beta_2)$ is a good deal slower. It would of course have been possible to develop the series to further terms by retaining semi-invariants of higher order, but even if for n less than 50 the resulting series were found to converge, it seems likely that the test in such cases would be of no great practical value. The test might enable us to say that in a sample of 20, let us suppose, values of $\beta_1=\cdot 8$ and $\beta_2=4\cdot 8$ could well have occurred in random sampling from a normal population. But such a sample *might* have come from a population in which, let us say, $\beta_1=1\cdot 5$, $\beta_2=5\cdot 5$; hence the sample data alone would give us no confidence in assuming normality in the population. Such an assumption must be justified from outside evidence which the sample values, while they would not contradict, would hardly strengthen.

An exact solution of the problem must await a knowledge of the true sampling distributions of $\sqrt{\beta_1}$ and β_2. In the meantime an approximate solution of some practical value can be obtained. Consider first the distribution of $\sqrt{\beta_1}$. Table I shows that this is a symmetrical leptokurtic curve which tends fairly rapidly to the normal. Fisher has obtained an approximation to its probability integral by constructing a function of x which as far as terms in n^{-2} is normally distributed with unit standard deviation†. The relation between this function, ξ, and x is given as follows:

$$\xi = x\left(1+\frac{3}{n}+\frac{91}{4n^2}\right)-\frac{3}{2n}\left(1-\frac{111}{2n}\right)(x^3-3x)-\frac{33}{8n^2}(x^5-10x^3+15x) \quad \ldots(24),$$

the coefficients being determined so that $\kappa_2(x)$, $\kappa_4(x)$ and $\kappa_6(x)$ (or $\mu_2(x)$, $\mu_4(x)$ and $\mu_6(x)$) are correct as far as terms in n^{-2}. But the expression used for $\kappa_6(x)$, namely $15120/n^2$, containing only a single term is of doubtful value as an approximation to the 6th semi-invariant of x even at $n=100$. To proceed by this method to terms in n^{-3} would involve the calculation of $\kappa_8(x)$. It seems therefore possible that as good an approximation will be obtained by assuming that the distribution of $\sqrt{\beta_1}$ may be closely represented by a Type VII curve of form

$$y = y_0(1+(\sqrt{\beta_1})^2/a^2)^{-m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(25),$$

whose constants are to be determined from the values of $\sigma_{\sqrt{\beta_1}}$, and $B_2(\sqrt{\beta_1})$ given in (10) and (11) above. Table I suggests that for $n \geqslant 50$ the expression (10) is completely adequate, while little error will be involved in using (11) for $n \geqslant 75$. For a curve of Type VII with the moments of (10) and (11) it can be shown that

$$\beta_4 = \frac{\mu_6}{\mu_2^3} = 15\left(1+\frac{36}{n}-\frac{288}{n^2}+\ldots\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots(26).$$

* These approximations to the standard errors were first given by K. Pearson in 1901, *Phil. Trans.* Vol. 198 A, p. 278.

† *Loc. cit.* pp. 233—235.

On the other hand, using Fisher's value for $\kappa_6(x)$ quoted above, the true β_4 of the x-distribution as far as terms in n^{-2} is

$$\beta_4 = 15\left(1 + \frac{36}{n} + \frac{144}{n^2} + \ldots\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (27).$$

At $n = 100$ the error is about 4 °/₀ which will probably not affect the form of the curve seriously in the region of significant frequency. In Table III are compared at $n = 50$ and 100 the chances P_1, P_2 and P_3 of $\sqrt{\beta_1}$ exceeding in sampling certain multiples of its standard error, $\sigma_{\sqrt{\beta_1}}$, found on three different hypotheses, namely:

(1) that $\sqrt{\beta_1}$ is normally distributed with $\sigma_{\sqrt{\beta_1}}$ given by (10),

(2) that $\sqrt{\beta_1}$ follows the Type VII curve,

(3) that the ξ of (24) is normally distributed with unit standard deviation, where $x = (n-1)\sqrt{\beta_1/6(n-2)}$.

TABLE III. *Approximations to Probability Integral of* $\sqrt{\beta_1}$.

Values of $\sqrt{\beta_1}/\sigma_{\sqrt{\beta_1}}$	$n=50$			$n=100$		
	P_1	P_2	P_3	P_1	P_2	P_3
1·2	·1151	·1094	·1159	·1151	·1113	·1125
1·6	·0548	·0534	·0532	·0548	·0539	·0535
2·0	·0228	·0241	·0205	·0228	·0237	·0227
2·4	·0082	·0102	·0067	·0082	·0096	·0087
2·8	·0026	·0042	·0020	·0026	·0037	·0032

The values required on hypothesis (2) have been found by interpolating in "Student's" Tables*. For $n = 50$ a value of $B_2(\sqrt{\beta_1}) = 3{\cdot}45$ was used†. It will be seen that at $n = 100$ the differences are of very little practical importance; at $n = 50$, although they are larger, it seems impossible to say without further information whether P_2 or P_3 is the more accurate. There would be little error involved in assuming the distribution of $\sqrt{\beta_1}$ to be normal with $\sigma_{\sqrt{\beta_1}} = \sqrt{6/n}$ for $n \geqslant 100$. In the Table printed at the end of this paper giving the 5 °/₀ and 1 °/₀ points for different values of n, I have, however, assumed the distribution to be of Type VII with the moments given by (10) and (11). The deviations from the mean to the ordinates cutting off these tail areas were found with the help of "Student's" Tables and graphical interpolation.

The distribution of β_2 is less easy to deal with. Fisher has suggested the use of another normally distributed function, ξ, of the x of equation (12), but as the transformation depends upon the use of expressions for $\kappa_4(x)$ and $\kappa_5(x)$ containing each only the first term of an expansion in inverse powers of n, the degree of

* *Metron*, Vol. v. 3, 1925, pp. 113—120.

† As far as the term in n^{-3}, the value shown in Table I is 3·4712, and a rough guess at the effect of the term in n^{-4} was made.

accuracy of the method is very uncertain. If the values of $B_1(\beta_2)$ and $B_2(\beta_2)$ given in Table II be plotted it will be found that they fall on a curve in the Type IV area which converges on the Type V and Type III lines, slowly approaching the Normal Point as $n \to \infty$. I have therefore made the assumption that the distribution of β_2 can be approximately represented by a Pearson Type IV curve with the moment coefficients given by the expressions (20)—(23), that is to say, by an equation of form

$$y = y_0 \left(1 + \frac{x^2}{a^2}\right)^{-p} e^{-\nu \tan^{-1} \frac{x}{a}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (28).$$

At $n = 100$ it will be seen from Table II that the four terms in the expansion for $B_2(\beta_2)$ are not sufficient to insure convergence even to the first decimal place, but for $B_1(\beta_2)$ they are so. Experience in curve fitting suggests, however, that this degree of uncertainty in B_2 when B_1 is known is not likely to have much influence on the deviation from the mean to the 5 °/₀ and 1 °/₀ probability points. That is to say, the chief danger of error present at $n = 100$ will not be due to uncertainty as to the B_2 of the Type IV curve, but to the use of a Type IV curve at all in place of the true curve. What this degree of approximation may be cannot at present be judged; Pearson curves based on theoretical values of the first four moment coefficients have been found in other problems to provide very satisfactory approximations to sampling distributions*, but these skew leptokurtic systems form a somewhat extreme case. It may, however, be said without hesitation that the results set out in Table IV below provide a test for normality of β_2 which will be far more accurate than has hitherto been available. $\sqrt{24/n}$ is a good approximation to the standard error of β_2 at $n = 50$, but even at $n = 1000$ the sampling distribution is not normal, viz. $B_1 = \cdot 21$, $B_2 = 3 \cdot 52$.

The method of computation was as follows. The ordinates of Type IV curves with B_1 and B_2 as in (22) and (23) were calculated for the cases $n = 100$, 150, 250 and 1000. From these were obtained by quadrature the deviations from the mean in terms of the standard deviation to the ordinates cutting off 5°/₀ and 1°/₀ tail areas. As n increases these deviations tend to the normal curve values of $\pm 1 \cdot 6449$ and $\pm 2 \cdot 3263$ respectively. Also as n increases the deviations approach closely the corresponding deviations in the Type III curve which has the same value of B_1; these latter were found from the Tables of the Incomplete Gamma Function. With these results to form a guide, it was possible to obtain graphically with sufficient accuracy the deviations from the mean to the 5 °/₀ and 1 °/₀ points for all the other Type IV curves required†. These deviations with the appropriate means and standard deviations given by (20) and (21) have provided the limiting values of β_2 given in Table IV.

* E.g. when used with experimental data, in connection with the distributions of the mean and the variance. A. E. R. Church, *Biometrika*, Vol. XVIII. pp. 321—394. Or again when compared with true theoretical curves as for the sampling distributions of p_{11}. Pearson, Jeffery and Elderton, *Biometrika*, Vol. XXI. pp. 164—201.

† The error involved in this process should not be greater than a single unit in the last (2nd) decimal place of the values of β_2 tabled. This can hardly be *greater* than the error involved in the assumption that Type IV curves will represent the sampling distribution of β_2.

TABLE IV. *5 °/₀ and 1 °/₀ Points for $\sqrt{\beta_1}$, β_1 and β_2.*

Size of Sample	$\sqrt{\beta_1}$		β_1		β_2			
	Lower and Upper Limits		Upper Limits		Lower Limits		Upper Limits	
	5 °/₀	1 °/₀	10 °/₀	2 °/₀	1 °/₀	5 °/₀	5 °/₀	1 °/₀
50	·533	·787	·285	·619	—	—	—	—
75	·445	·651	·198	·424	—	—	—	—
100	·389	·567	·152	·321	2·18	2·35	3·77	4·39
125	·350	·508	·123	·258	2·24	2·40	3·70	4·24
150	·321	·464	·103	·216	2·29	2·45	3·65	4·14
175	·298	·430	·089	·185	2·33	2·48	3·61	4·05
200	·280	·403	·078	·162	2·37	2·51	3·57	3·98
250	·251	·360	·063	·130	2·42	2·55	3·52	3·87
300	·230	·329	·053	·108	2·46	2·59	3·47	3·79
350	·213	·305	·045	·093	2·50	2·62	3·44	3·72
400	·200	·285	·040	·081	2·52	2·64	3·41	3·67
450	·188	·269	·035	·072	2·55	2·66	3·39	3·63
500	·179	·255	·032	·065	2·57	2·67	3·37	3·60
550	·171	·243	·029	·059	2·58	2·69	3·35	3·57
600	·163	·233	·027	·054	2·60	2·70	3·34	3·54
650	·157	·224	·025	·050	2·61	2·71	3·33	3·52
700	·151	·215	·023	·046	2·62	2·72	3·31	3·50
750	·146	·208	·021	·043	2·64	2·73	3·30	3·48
800	·142	·202	·020	·041	2·65	2·74	3·29	3·46
850	·138	·196	·019	·038	2·66	2·74	3·28	3·45
900	·134	·190	·018	·036	2·66	2·75	3·28	3·43
950	·130	·185	·017	·034	2·67	2·76	3·27	3·42
1000	·127	·180	·016	·032	2·68	2·76	3·26	3·41
1200	·116	·165	·013	·027	2·71	2·78	3·24	3·37
1400	·107	·152	·012	·023	2·72	2·80	3·22	3·34
1600	·100	·142	·010	·020	2·74	2·81	3·21	3·32
1800	·095	·134	·009	·018	2·76	2·82	3·20	3·30
2000	·090	·127	·008	·016	2·77	2·83	3·18	3·28
2500	·080	·114	·006	·013	2·79	2·85	3·16	3·25
3000	·073	·104	·005	·011	2·81	2·86	3·15	3·22
3500	·068	·096	·005	·009	2·82	2·87	3·14	3·21
4000	·064	·090	·004	·008	2·83	2·88	3·13	3·19
4500	·060	·085	·004	·007	2·84	2·88	3·12	3·18
5000	·057	·081	·003	·006	2·85	2·89	3·12	3·17

(5) *Illustration of Use of Table IV.*

In a sample of 500 the following values are found:

$$\sqrt{\beta_1} = -\cdot 2040; \quad \beta_1 = \cdot 0416; \quad \beta_2 = 3\cdot 7823.$$

Is it possible that the sampled population was normal?

The table shows that in 5 °/₀ of random samples from a normal population $\sqrt{\beta_1}$ may be expected to be less than $-\cdot 179$, and in 1 °/₀ less than $-\cdot 255$. The observed value falls in between these limits. In using β_1 positive and negative

values of $\sqrt{\beta_1}$ are clubbed together, and we see that in 10 °/₀ of random samples β_1 may be expected to be greater than ·032 and in 2 °/₀ greater than ·065. The observed value of course falls again between the limits. For β_2 we see that only 1 °/₀ of samples can be expected to give a β_2 greater than 3·60; the observed value of 3·7823 lies outside the limit. The test therefore provides a doubtful answer when applied to β_1 but a decisive one when applied to β_2, and we may conclude that it is practically certain that the sample has not been drawn randomly from a normal population.

(6) *Summary.*

(*a*) The work of Fisher and of Craig has made it comparatively simple to obtain expressions for the moment coefficients of the distributions of $\sqrt{\beta_1}$ and β_2 in samples of n from a normal population. These expressions are in the form of series in inverse powers of n. In order to see more clearly the degree of convergence of these series and to obtain more accurate values in smaller samples, it was necessary to extend the series beyond the point reached by those writers. This it has been possible to carry out with the aid of new results obtained by Wishart.

(*b*) The moment coefficients show that the distribution of $\sqrt{\beta_1}$ is a symmetrical leptokurtic curve which tends to the normal fairly rapidly as n increases. For rough purposes it may be taken as normal with a standard error of $\sqrt{6/n}$ for $n \geqslant 100$. The distribution of β_2 is an extremely skew curve at $n = 100$, and even when $n = 1000$ can hardly be considered as normal.

(*c*) A table has been given of the approximate 5 °/₀ and 1 °/₀ probability points for $\sqrt{\beta_1}$ and β_2, based on the assumption that the true distribution may be adequately represented by Pearson curves with the correct first four moment coefficients. This table starts at $n = 50$ for $\sqrt{\beta_1}$ and $n = 100$ for β_2.

(*d*) A complete test for normality must really be two-sided; it must help us not only to determine whether the population sampled could have been normal, but also to judge how far from normal the population might have been. A knowledge of the true sampling distribution of $\sqrt{\beta_1}$ and β_2, when the population is normal, would enable us to answer the first point however small the sample may be, but not the second point.

(*e*) It is to be hoped that the true sampling distributions will be found, not only to remove any doubt as to the accuracy of the test, but also for the light that will be thrown on the adequacy of these methods of approximation—information that will be of considerable value in handling similar problems in the future.

MISCELLANEA.

An unusual Frequency Distribution—The Term of Abortion.

By THOS. VIBERT PEARCE, F.R.C.S. England.

Abortion in women is becoming more prevalent, and gradually will assume economic and political importance. From being a purely medical problem it will gather biological and chemical interest, since the interlocking of the female reproductive hormones is slowly being laid bare.

Out of 300 women admitted to St Giles' Hospital who left the hospital following completed abortion, 283 patients were able to give enough information to allow of a fairly reliable estimate of the term of gestation prior to abortion. Term of abortion when used in this present connection does not mean the same thing as the time of developement of the foetus. Even if the time of insemination is known—quite an unusual piece of information—the time of impregnation is quite unknown, and it is difficult to see how it can ever be ascertained. Impregnation "may be postponed for days or possibly three or four weeks." In the absence of careful measurement of the foetus and frequently in the absence of the foetus itself, this possible lapse of time between insemination and impregnation compels consideration of the term of abortion from the standpoint of the maternal partner in this pathological condition. Abortion is commonly regarded as a disease of the mother and not as a disease of the foetus, although there is no logical or objective support for that opinion. Incidentally it is quite possible there may be a type of abortion due to defect in the paternal germ plasm. At any rate the mother seems to be the more important sufferer, and the post-menstrual term in default of a better definition is used as the term of abortion. By plotting the frequencies of the post-menstrual term some light might be thrown on the likelihood of abortion occurring at the expected times of menstruation which are masked or abolished by impregnation.

The women could generally remember the date of their last menstruation, although for some obscure reason they found it quite difficult to forecast the date of the next. Frequently they were rather surprised at any attempt to find the exact inter-menstrual period. Some say it always occurs on the same day of the month, and has done so for months or years. They are then really claiming that the time of menstruation is sometimes governed by the calendar fixed by Pope Gregory—a pretension that is hardly convincing. Some women say that their menstrual cycle lasts exactly four weeks, but yet are ignorant of the day of the week on which it commences. The results of a sympathetic and veiled cross-examination really suggest that quite a large proportion of these women were ignorant of the length of the menstrual period beyond their estimate that it lasts "a month." Once a woman naïvely referred me to her husband. A rather young married woman told me to ask her mother, because she always menstruated concurrently with her. One patient was rather sorry for herself, for she unfortunately commenced menstruating on washing day, which was Monday—good evidence that she had a 28-day cycle. It was hoped that some estimate of the variability of the menstrual period might be got from the statements of these women. The statements hardly ever bore examination, and were abandoned as being hopelessly unreliable. These women who had aborted or were aborting did not seem more stupid than the generality of women. Certainly their period of amenorrhoea does seem long enough for them to have forgotten their proper menstrual periods.

Besides the commencement of the last menstruation, the other point of time that is fairly satisfactorily remembered is the date of the passage of the foetus, or, in default of which, the

dates of the maximum pain and bleeding which, if they happen to coincide, fix very well the date of abortion. The commencement of symptoms before abortion is difficult to date, and hence has not been used to calculate the term of abortion. It is very hard to disentangle post-impregnation menstruation from the symptoms of abortion. Term of abortion is therefore defined as the number of days between the commencement of the last menstrual period and the passage of the foetus.

For diagrammatic purposes the term of abortion has been plotted in nearest weeks. Division by 7 is very convenient and leads to no awkward half-divisions. The question of abortion at the expected menstrual times would be much better tackled by dividing the term in days by the number of days of the patient's own menstrual cycle. A frequency diagram of term of abortion along a scale of menstrual months could then be made. Such a diagram for these women at any rate would be unreliable. This is very disappointing, for it was rather hoped that by expressing the term of abortion in menstrual months it might be possible to get some evidence which would help to decide whether the abortion was spontaneous or artificial. Presumably spontaneous abortion would occur at an expected menstrual time, while induced abortion would occur after a menstrual period had been missed and the assault on the pregnancy would be renewed after the missing of the next expected menstruation.

The variability of the menstrual period in different women when compared with the optimum term for abortion will not cause the frequency curve of the term of abortion to give so little help as it might on this question of abortion at the expected menstrual times. The mean term of abortion is about 13·41 weeks or 3 calendar months, and the variability of menstrual period seldom exceeds 1 or 2 days on either side of 30 days. It is only in the later months that the error of 1 or 2 days would be multiplied to amount to a week. By reference to the table, the plateau at the 17th and 18th weeks does suggest that this kind of error has occurred there. The 17th and 18th weeks may include abortions that occurred at the 4th month of missed menstruation. On a scale of menstrual months, the frequencies for these weeks might be amalgamated.

Frequency Table of Term of Abortion.

Term (in weeks)	4	5	6	7	8	9	10	11	12	13	14	15	16
Afebrile	3	6	7	9	11	20	13	16	9	17	9	12	6
Febrile	0	1	3	4	3	9	9	5	9	11	7	7	4
Total	3	7	10	13	14	29	22	21	18	28	16	19	10

17	18	19	20	21	22	23	24	25	26	27	28	Total	Mean Term	Standard Devn.
7	8	7	2	1	4	2	2	2	4	4	1	182	13·08	5·57
6	6	1	2	1	6	2	2	1	0	2	0	101	14·00	5·03
13	14	8	4	2	10	4	4	3	4	6	1	283	13·41	5·41

To the writer the frequency diagram does suggest that abortion occurs especially at the menstrual times, even allowing for the human characteristic of rationalisation both in patients and the observer. Whether there is any statistical warrant for such opinion seems to be a difficult problem. Presumably it would be necessary to test the goodness of fit of a frequency curve which admits of a series of maxima at regular intervals occurring along a curve which itself mounts to an apex about its mid-point. Perhaps another way of treating the problem

would be one of dissection. The scale is one of 7 months. On the hypothesis of abortion being more frequent at the menstrual times, 7 summits to the curve could be postulated. The whole curve is then regarded as made up of the summation of 7 very pointed normal frequency curves. Such a method implies that abortion at any one month is a different "clinical entity" from abortion at any other. Such an argument could not be convincing, for women who have had multiple abortions do not miscarry at the same term every time. In fact the impression left is the reverse. A normal curve would not at all represent the terms of abortions of a patient who has had multiple abortions.

[A periodogram analysis of the above data shows, as far as weekly ranges will permit, a period of four and a half weeks. Ed.]

Socrates. And furthermore, the midwives, by means of drugs and incantations, are able to arouse the pangs of labour and if they wish, to make them milder, and to cause those to bear who have difficulty in bearing; and they cause miscarriages if they think them desirable.

Theaetetus. That is true.

Socrates. Well, have you noticed this also about them, that they are the most skilful of matchmakers, since they are very wise in knowing what union of man and woman will produce the best possible children?

Theaetetus. I do not know that at all.

Plato, *Theaetetus*
(Loeb Classical Library: H. N. Fowler).

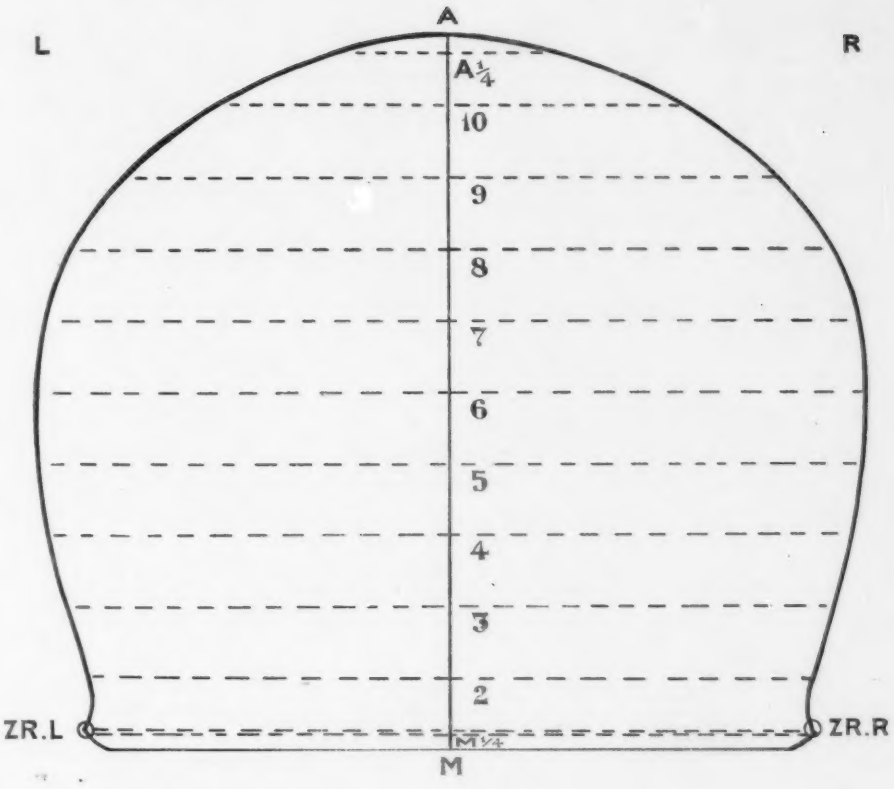


FIG. I. Transverse Type Contour of 39 ♂ Egyptian Skulls from Sedment.

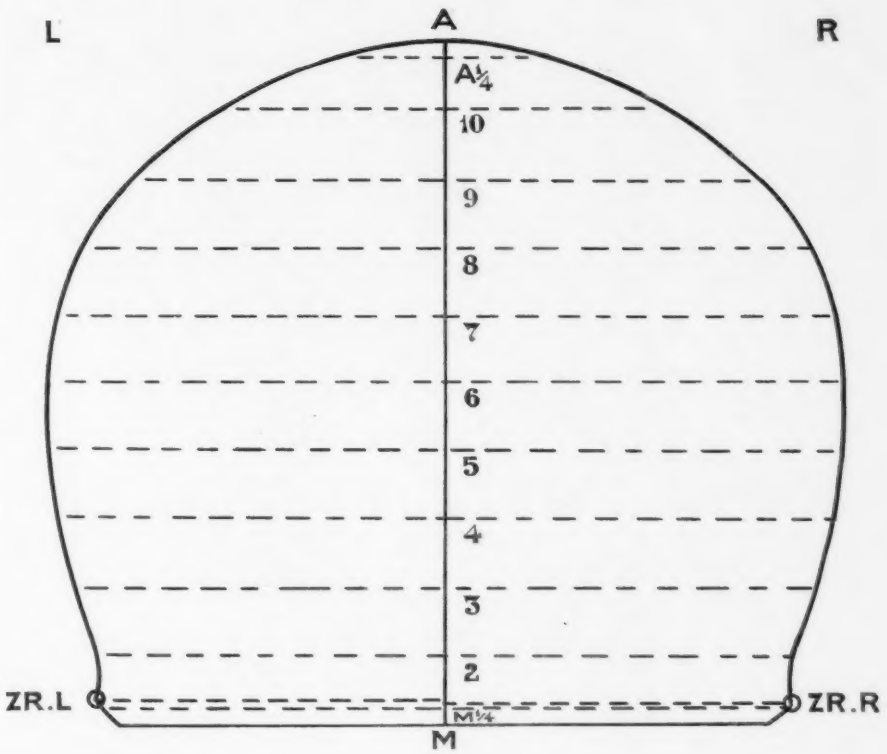


FIG. II Transverse Type Contour of 30 ♀ Egyptian Skulls from Sedment.

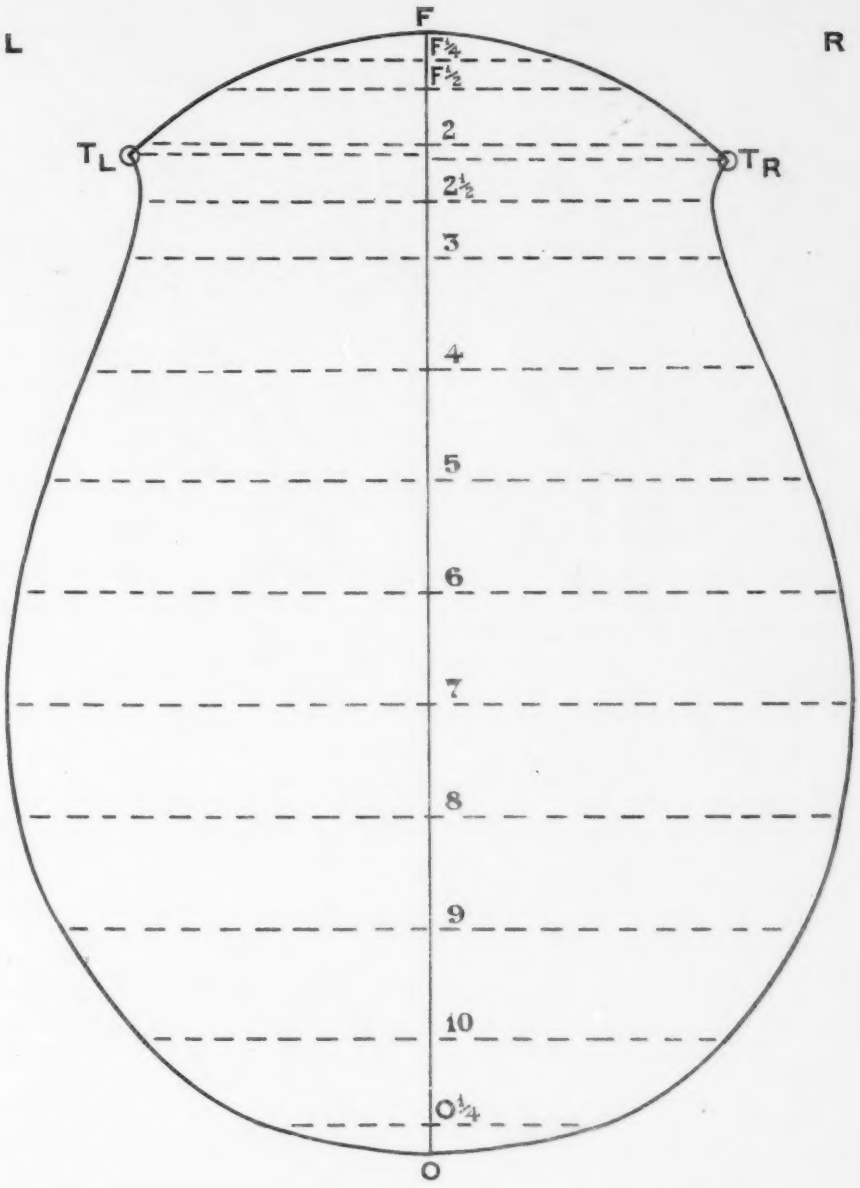


Fig. III. Horizontal Type Contour of 38 ♂ Egyptian Skulls from Sedment.

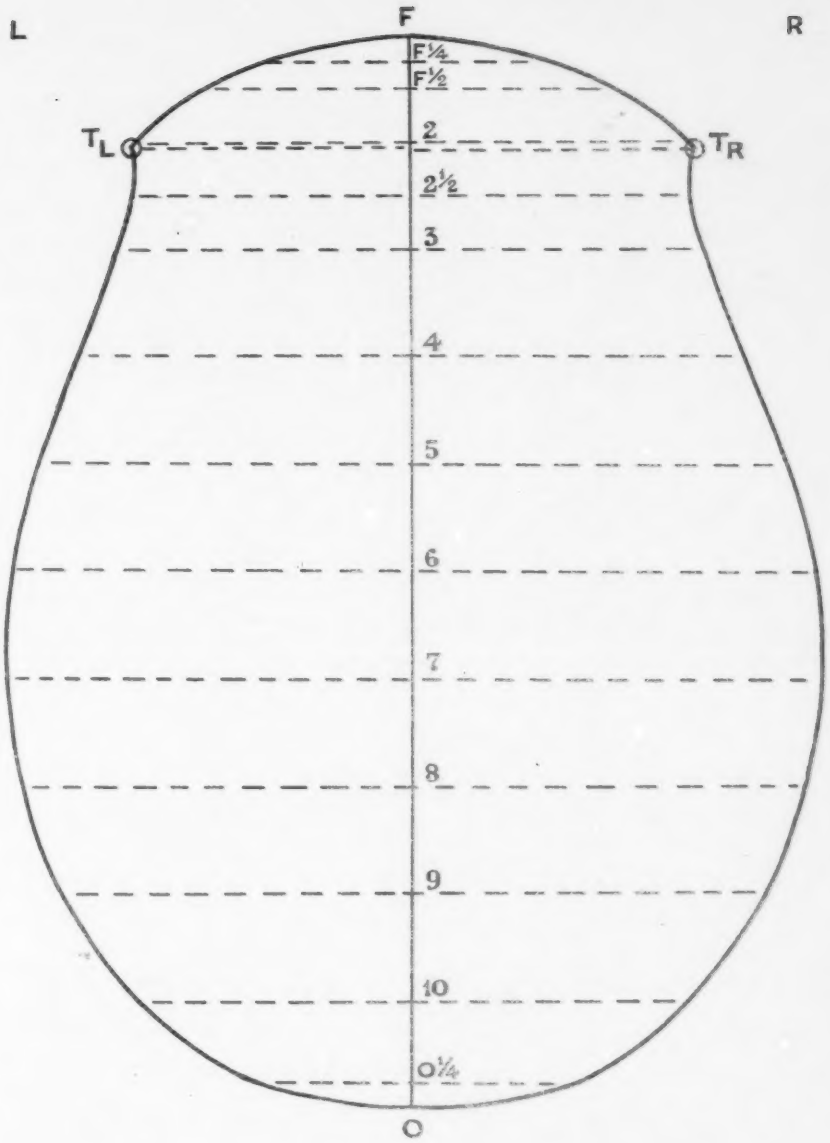


FIG. IV. Horizontal Type Contour of 30 ♀ Egyptian Skulls from Sedment.

Bio

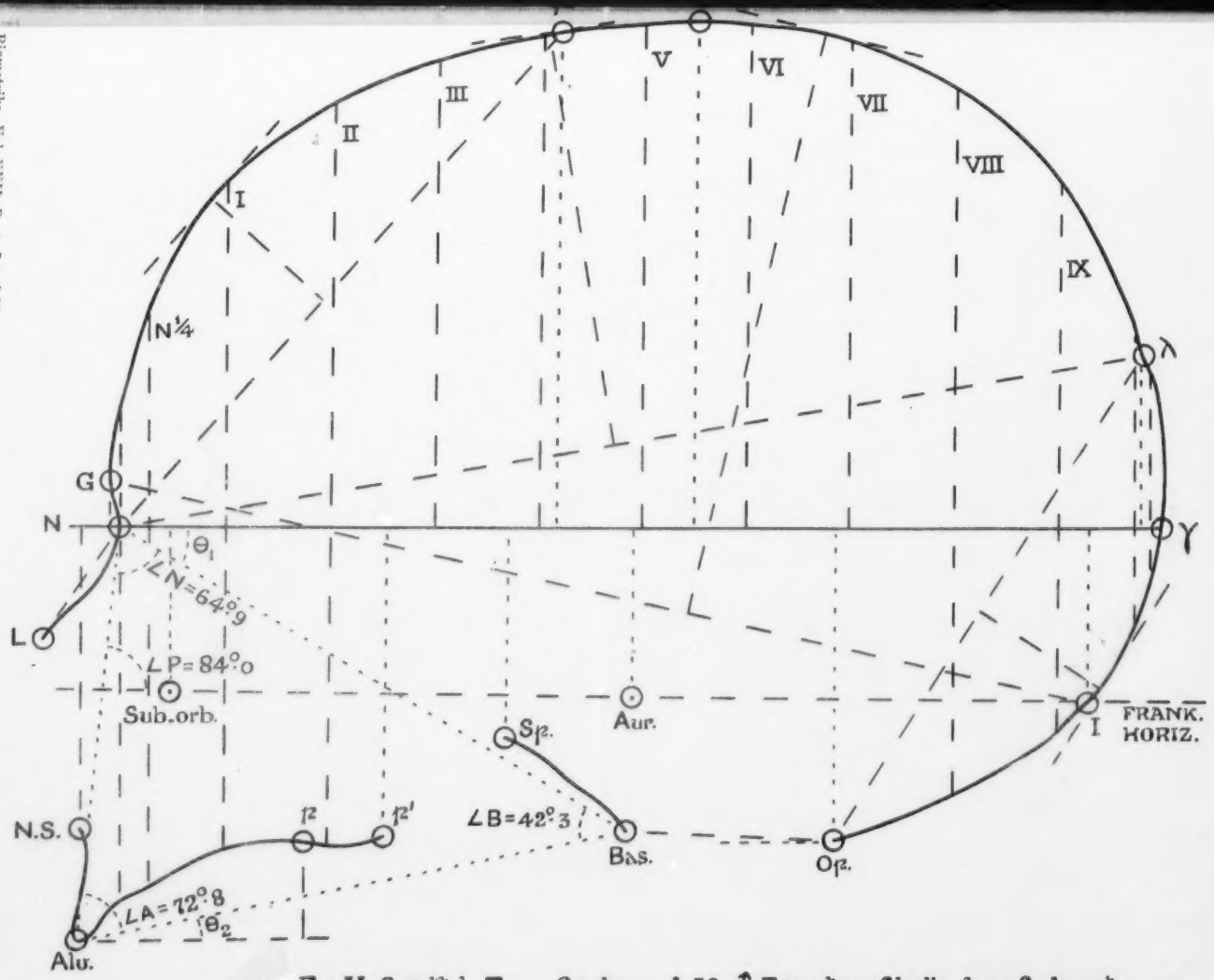
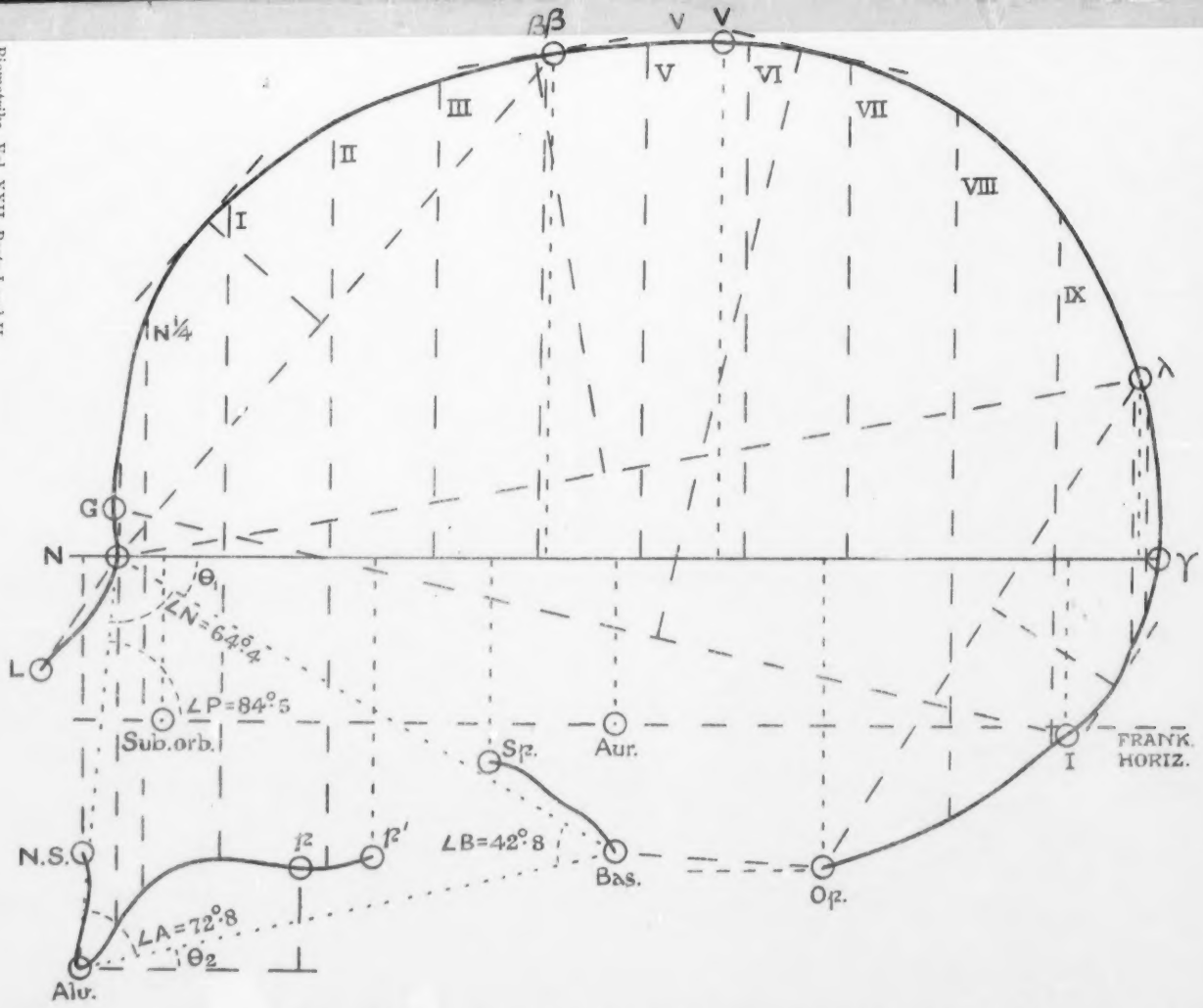


FIG. V. Sagittal Type Contour of 39 ♂ Egyptian Skulls from Sedment.



VI. Sagitta[illegible]e Contour of 30 ♀ Egyptians Skulls from Sedmeni.

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BIOMETRIKA. Vol. XXII, Parts I and II

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